

Time- and price-based product differentiation in hybrid distribution with stockout-based substitution

Ramzi Hammami ^{1*}, Erfan Asgari ², Yannick Frein ², Imen Nouira ¹

¹ Rennes School of Business, 2 rue Robert d'Arbrissel, 35065, Rennes, France.

² Univ. Grenoble Alpes, CNRS, Grenoble INP, G-SCOP, F-38000 Grenoble, France.

hammami.ramzi@gmail.com, erfan.asgari@grenoble-inp.fr, yannick.frein@grenoble-inp.fr, imen.nouira@rennes-sb.com

* Corresponding author

Abstract. A delivery mix that includes delivery from stock and drop-shipping is of interest to many internet retailers. We consider a retailer serving a time- and price-sensitive market with two substitutable products that differ in the guaranteed delivery time and price, an express product (delivered from the stock) and a regular product (drop-shipped). In case of stockout, customers may switch from the express product to the regular product. We study how to differentiate the products in terms of delivery times and prices and how to determine the stock level to maximize the retailer's expected profit while satisfying service constraints. We solve different variants of the problem and derive insights into the optimal retailer's strategy. In addition, we study the impact of stockout-based substitution. This paper is the first to investigate time- and price-based differentiation along with inventory decisions for a retailer who relies on a hybrid distribution to satisfy a time- and price-sensitive demand subject to stockout-based substitution. When prices and stock are fixed, in addition to minimum and maximum time differentiations, a medium differentiation strategy may be optimal but depends on the stock level. When only prices are fixed, there exists a price differentiation limit below which a minimum time differentiation is optimal, and above which only the express product should be offered. For the general model, numerical experiments show that a higher stockout-based substitution leads to greater time differentiation (which is consistent with the results of previous models) and more stock. However, this would not impact the price differentiation.

Keywords: Supply chain management; Time- and price-based differentiation; Delivery time quotation; Dual-channel retailing; Stockout-based substitution.

1. Introduction

Many retailers rely on a delivery mix that includes the traditional delivery from stock (DFS) and the drop-shipping to satisfy demand. With drop-shipping, a retailer simply forwards customer orders to manufacturers or wholesalers who fulfill those orders directly for a predetermined price to be paid by the retailer (Khouja and Stylianou 2009). Drop-shipping has some advantages for the retailer, including savings in holding cost, but drop-shipping typically incurs a longer delivery time (DT), which may deter time-sensitive customers (Khouja 2001, Khouja and Stylianou 2009, Rabinovich et al. 2008). DFS implies a shorter and more reliable DT, but incurs a holding cost that may increase with uncertainty. A mix of DFS and drop-shipping has proven to be an efficient distribution strategy, particularly when there is a high uncertainty in demand and/or

replenishment lead time. In such cases, the drop-shipping provides the retailers with the option to reserve the stock internally for high-priority orders while fulfilling regular orders through drop-shipping (Ayanso et al. 2006).

With a mix of DFS and drop-shipping, a product is offered with two varieties that have different DTs and can generate different margins (Ayanso et al. 2006). Rabinovich et al. (2008) studied the handbag and luggage segment of online retailing (e-Bags.com) and proved empirically that an internet retailer can obtain higher margins when it is willing and able to promise a shorter DT. Over the last Christmas holidays, Cdiscount.com (a French leading online retailer), offered customers two options to buy some products. In the first option, the product is immediately delivered from Cdiscount store in France. In the second option, the product is delivered directly from the supplier in China. This second option is less expensive for the customer but leads to a much longer and less reliable DT. Two real examples are presented below.

		
Option 1. Delivery from Cdiscount store	Delivery: 21-22 November Price: €179.99	Delivery: 14-15 November Price: €10.13
Option 2. Delivery from Chinese supplier	Delivery: 05-19 December Price: €169.99	Delivery: 13-17 December Price: €7.99

Figure 1. Real examples of products delivered with a mix of DFS and drop-shipping on Cdiscount.com

The case where the retailer controls this hybrid distribution mode and decides how to differentiate the products with respect to DT and price is relevant to powerful distributors and retailers (Li et al. 2019, Tian et al. 2018, Hagi and Wright 2015). Practical illustrations are given in the following examples. Heydari et al. (2019) provided the example of Digikala, the most famous and successful online retailer in Iran (www.Digikala.com). Digikala owns both DFS and drop-shipping channels. As a result, Digikala offers substitutable products through a dual-channel structure and decides the characteristics of both channels. In the healthcare sector, a global leader in clinical diagnostics and industrial microbiology distributes in vitro diagnostic medical devices while offering two possibilities to customers. The first (premium) alternative is the delivery from stock (an advanced stock is located close to demand zone). The second (standard) alternative consists in forwarding the order to the manufacturing site; the product is then received at the distribution center and delivered to the customer with a longer DT and a lower price. There are also many examples that can be found on Amazon.com. Amazon offers substitutable goods that can be either drop-shipped from upstream suppliers or shipped directly from Amazon fulfillment centers (Ayanso et al. 2006, Rabinovich et al.

2008). For instance, an electrical grill may be delivered from Amazon store or drop-shipped from the supplier “Truthful Life” (www.Amazon.com). Truthful Life offers a saving of \$9.99 but the DT is 11 days longer. So, consumers who are willing to wait 11 days save the \$9.99. Giant retailers like Amazon have the power to determine which suppliers to include on their platform and impose the use of adequate logistics service which enable to guarantee a desired DT to customers. The above examples (a same company that manages both production and distribution or a giant retailer that can impose his conditions to the suppliers) correspond to cases where the retailer (distributor) acts as a centralized decision-maker and decides how to differentiate express and regular products.

Following the above practical examples, we consider the case of a retailer who relies on a hybrid distribution mode with a mix of DFS and drop-shipping to sell two substitutable products (regular and express), where the regular product refers to the drop-shipped product, and the express product refers to the product delivered from stock. In compliance with the above examples, the express product has a shorter DT but a higher price than the regular product. Figure 2 illustrates the studied supply chain (SC).

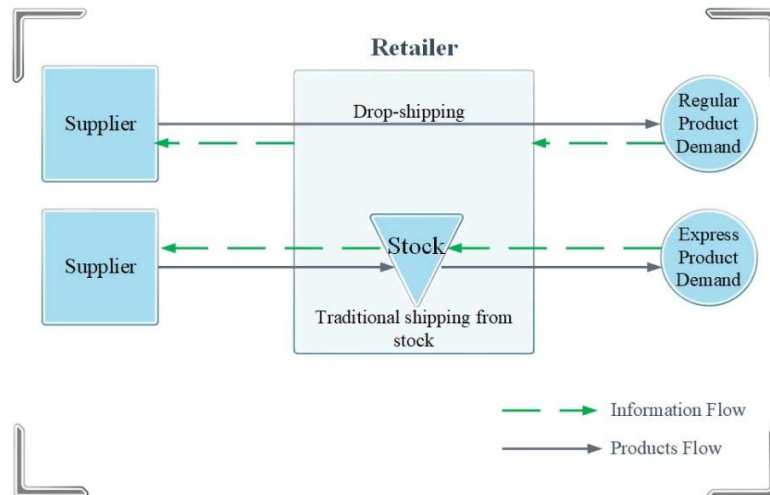


Figure 2. Illustration of the studied hybrid system with a mix of DFS and drop-shipping

The DT of the drop-shipped product and the price of the express product cannot be chosen without considering the other product because of the potential cannibalization between the two products. Therefore, it is critical for the retailer to strategically differentiate the two products to achieve the maximum profit. This study investigates the DT, price, and inventory decisions faced by a retailer that uses a mix of DFS and drop-shipping to serve a time- and price-sensitive market with two substitutable products. The products differ in terms of the guaranteed DT and price. The customers observe the pair of DT and price quoted by the retailer for each product and make their purchase decisions. In case of stockout, some customers initially interested in the express product may switch to the regular product, whereas others may leave the system (lost sales). Many authors have provided practical examples to illustrate how customers can be served with drop-shipping in case of stockout (e.g., Khouja and Stylianou 2009, Ayanso et al. 2006). Stockout-based substitution (S-b-S) impacts the demand of both express and regular products. Therefore, it is important to consider the S-b-S and

study its effect on DT, price, and inventory decisions. We examine this problem when the demand process is stochastic, inventory replenishment lead time is stochastic, and drop-shipping lead time is also stochastic. The retailer, acting as a centralized decision-maker, decides how to differentiate the products (with respect to DT and price) and determines the stock level while satisfying the service level constraint for each product. The products are substitutable, so the demand for each product decreases in its price and quoted DT and increases in the price and quoted DT for the competing product. In addition, as we consider the S-b-S, the effective demand of each product depends also on the stockout probability, which leads to a non-linear demand function that is sensitive to DT, price, and stock level.

There are several trade-offs in the model. The express product can be sold with a higher margin but requires holding inventory, which implies an additional cost. If the express product is profitable, it may be interesting to hold more stock than the minimum level imposed by the service constraint as this reduces the stockout probability and allows to serve more express customers. To increase the demand for the express product, the retailer can quote a longer DT for the regular product. However, an increase in the express demand may require holding more inventory to satisfy the service level constraint. The S-b-S is expected to have a significant impact on such trade-offs. Additionally, with the presence of S-b-S, it may be interesting to hold less stock (to reduce the inventory cost) although this implies some loss of express demand, since this loss will be partially transformed into additional demand for the regular product. However, this may lead to a longer waiting time in the drop-shipping channel and, consequently, a higher risk of violating the service level constraint.

To provide insights into these trade-offs, we study different variants of the problem: (i) The retailer decides the DT differentiation where the stock and prices are fixed. (ii) The retailer decides the stock level and the DT differentiation where the prices are fixed. (iii) The retailer decides the DT and price differentiations as well as the stock level (general model). In the first setting, we determine the closed-form expression of the optimal solution. We show analytically that it is not always optimal to adopt an extreme time differentiation strategy and that an intermediate strategy can be more valuable but should be adjusted in function of the stock level. In the second setting, we determine the optimal solution when S-b-S is ignored and provide a near-optimal solution in the presence of S-b-S. In the case without S-b-S, analytical results show that there exists a threshold price below which it is optimal to offer both products with the smallest possible time differentiation (i.e., offering the regular product with the shortest feasible DT) and above which only the express product should be offered. Considering the S-b-S, the profit function profile changes. We show numerically that quoting the DT without considering the effect of S-b-S leads to a considerable loss. Finally, using the analytical results obtained in the first two settings, we transform the general model into a quasi-equivalent single-variable model. Based on extensive numerical experiments, we show that a higher rate of S-b-S should normally lead to greater time differentiation and more stock. However, this would not impact the price differentiation.

There are valuable studies on time-based differentiation for make-to-order (MTO) manufacturers (e.g., [Pekgün et al. 2017](#), [Zhao et al. 2012](#), [Boyaci and Ray 2003, 2006](#)), and for dual-channel SCs where the manufacturer manages the direct channel (e.g., [Hua et al. 2010](#), [Modak and Kelle 2019](#)). There are also interesting studies on how retailers control the inventory with drop-shipping option (e.g., [Ayanso et al. 2006](#), [Khouja and Stylianou 2009](#), [Ma and Jemai 2019](#)). However, the problem studied in this paper has not been explored in the extant literature. We contribute by solving a new problem and providing new results. For instance, while [Boyaci and Ray \(2003, 2006\)](#) and [Hua et al. \(2010\)](#) showed that the market characteristics govern optimal product positioning and time differentiation strategies, we generalize this result and show that the S-b-S also plays an important role, particularly in low competitive environments that are typically characterized by high S-b-S rates. Our analysis sheds light on an important new finding as we show in all settings that a higher S-b-S must lead retailers to increasing the time differentiation. Ignoring S-b-S leads to undertaking sub-optimal decisions. Our work explains how the retailer's stock level impacts the time differentiation strategy. This analysis cannot be obtained from studies that focus on MTO manufacturing systems (e.g., [Boyaci and Ray 2003, 2006](#), [Zhao et al. 2012](#), [Pekgün et al. 2017](#)) or from studies on inventory models with drop-shipping but without DT differentiation decisions (e.g., [Ayanso et al. 2006](#), [Khouja and Stylianou 2009](#), [Ma and Jemai 2019](#)). Indeed, while an extreme time differentiation strategy is proved to be optimal in most related works, we find that when the stock level is greater than a given threshold value, a medium time differentiation strategy can be optimal. We also show that the higher the price differentiation is, the higher the time differentiation becomes, which confirms the findings of [Modak and Kelle \(2019\)](#) and [Boyaci and Ray \(2003, 2006\)](#). Then, we complement these results by showing that a higher price differentiation leads the retailer to holding more stock.

Section 2 gives an overview of the literature. In Section 3, we develop the general framework. In Sections 4, 5 and 6, we solve different variants of the model and derive analytical and numerical insights. Finally, in Section 7, we conclude and discuss the practical implications of our findings.

2. Related literature

The studies that come closest to our work are those considering a SC that offers two substitutable products differentiated in terms of DT and price. We discuss these studies and show how we contribute to the extant literature with respect to both modeling framework and managerial findings. Our work also relates to the inventory control literature under drop-shipping and, more generally, to the joint lead time quotation and pricing problem in time- and price-sensitive markets. We review relevant papers in these areas. A classification table that compares the different studies is provided at the end of this section.

2.1. Substitutable products differentiated in terms of DT and price

Most relevant works focus on the problem of an MTO manufacturer. [Boyaci and Ray \(2003\)](#) considered a firm selling two substitutable products to satisfy a linear price- and DT-sensitive demand. Demand is served

from two separate facilities, and each facility is modeled as an M/M/1 queue. In its general form, the problem is choosing the price of each product and the DT of the express product. They showed that considering capacity costs leads to increasing the time differentiation. [Boyaci and Ray \(2006\)](#) extended the previous work by considering the delivery reliability (i.e., minimum service level) as a decision variable. They found that the firm would offer the express product with a lower service level than the regular product when the market is balanced between time-sensitive customers and price-sensitive customers. [Zhao et al. \(2012\)](#) compared the uniform quotation mode, when a firm offers a single DT and price, to the differentiated quotation mode, when a firm offers a menu of DT and prices. They indicated that when DT-sensitive customers value a product no more than price-sensitive customers, a firm should use the uniform quotation mode. Our research differs from these studies in three key aspects. First, we consider the case where the product differentiation results from a mix of DFS and drop-shipping, whereas the above studies differentiate products by allocating a different production capacity to each one of them. Second, unlike the above studies, we incorporate inventory decisions which is relevant to the retailing context. Third, in our work, the switchovers are governed not only by price and DT disparities but also by the stockout.

There are also interesting studies on DT quotation and pricing in dual-channel SCs with one manufacturer and one retailer such as [Hua et al. \(2010\)](#) and [Modak and Kelle \(2019\)](#), or with two competing firms such as [Pekgün et al. \(2017\)](#). These studies, however, focused on a decentralized setting whereas we consider a centralized setting where all decisions are undertaken by a single retailer. In addition, the consideration of product differentiation and inventory decisions distinguish our work from the above studies.

With respect to the main findings, our work contributes to the research stream presented above by providing new results that allow to understand the interplay of inventory, DT quotation, pricing, and S-b-S. We examine how the retailer's inventory level impacts the DT-based differentiation and how the S-b-S impacts the price, DT and inventory level. For instance, we show that the retailers must account for the effect of S-b-S to avoid sub-optimal decisions. The existence of S-b-S leads the retailer to considering medium levels of time differentiation as a potential optimal strategy, whereas minimum or maximum time differentiations are often optimal in the existing models. We also find that a higher S-b-S leads to a greater time differentiation and more stock. However, this would not impact the price differentiation.

2.2. Inventory models with drop-shipping

[Khouja \(2001\)](#) first incorporated drop-shipping as an additional option into the news-vendor problem and found that drop-shipping significantly improves the expected profit. [Ayanso et al. \(2006\)](#) considered an online retailer facing two customer classes with respectively short DT needs satisfied via air-shipment and regular needs satisfied via ground-shipment. The authors identified the stock level below which low-priority orders are to be drop-shipped and in-house stock is to be exclusively reserved for high-priority orders. [Khouja and Stylianou \(2009\)](#) revisited the (Q, R) inventory model by using the drop-shipping option in case of a shortage. They showed that drop-shipping is more valuable when the replenishment lead time is long and the ordering

cost relative to the holding cost is small. [Ma and Jemai \(2019\)](#) used the news-vendor model to study the problem of a retailer who faces two types of demand: store demand and internet demand. Drop-shipping is used to fulfill internet demand if the available stock is not sufficient. The authors investigated the stock rationing level below which the retailer starts using drop-shipping. The above papers focused mainly on the impact of drop-shipping on inventory decisions. They proved the interest of offering the dual-channel option in addition to the traditional retail channel. Our study builds on this finding and investigates how to differentiate the product offered by each channel and how to adjust the stock level accordingly. Note that these works assumed exogenous demand, DT, and price and, thus, did not investigate the questions related to the product differentiation and S-b-S. Our work generalizes the above studies by showing that it can be optimal to offer two differentiated products where drop-shipping is used to satisfy the demand of the regular product, or to use drop-shipping only as a backup solution to serve the switching customers in case of stockout. In this latter case, the whole system can be viewed as a traditional shipping system with a combination of lost sales (customers who leave in case of stockout) and back-orders (customers who switch to drop-shipping in case of stockout).

2.3. Joint lead time quotation and pricing

In a pioneer paper, [Palaka et al. \(1998\)](#) studied the DT quotation, pricing, and capacity utilization for an MTO firm modeled as an M/M/1 queue and facing a linear price- and DT-sensitive demand. Using this framework, [Pekgün et al. \(2008\)](#) studied the decentralization of pricing and DT decisions where the production department quotes a DT and the marketing department quotes a price. [Liu et al. \(2007\)](#) studied a decentralized SC where the supplier, acting as a leader, decides the quoted DT and the wholesale price. The retailer, acting as a follower, decides the final price. [Albana et al. \(2018\)](#) extended the previous works by assuming that the production cost is not fixed but decreases in the quoted DT. [Hammami et al. \(2020a\)](#) proposed a new framework as they considered a two-stage SC with a DT decision associated with each stage and a final demand that depends on the overall DT quoted by the SC and the final price. This study is first to consider a tandem queue M/M/1-M/M/1 in the analysis of the DT quotation and pricing problem. [Hammami et al. \(2020b\)](#) investigated the policy of rejecting customers when the number of waiting orders is larger than a fixed threshold K . The authors showed that their proposed policy can be more profitable than the all-orders acceptance policy, and then studied the impact of K .

These studies examined the DT quotation and pricing for a single final product. Their findings cannot be generalized to investigate the case of substitutable products where it may be interesting to quote a longer DT or a higher price for a given product in order to favor the demand of the other product as we show in this paper. In addition, the above studies did not account for the inventory decisions and their impact on the DT quotation and pricing.

Table 1. Classification of the literature

Literature	Decisions	Demand	Decision-making	Main problem investigated	Modeling framework	Stockout
Boyaci & Ray (2003)	p, l	Linear $f(p, l)$ Substitution (2 varieties)	Centralized (MTO manufacturer)	Price-and time-based differentiation	Stochastic dual channel with (two M/M/1 queues in parallel)	No inventory
Boyaci & Ray (2006)	p, l, r	Linear $f(p, l, r)$ Substitution (2 varieties)	Centralized (MTO manufacturer)	price-, time- and reliability-based differentiation	Stochastic dual channel (two M/M/1 queues in parallel)	No inventory
Ayanso et al. (2006)	S	Exogenous (2 customer classes)	Centralized (retailer)	Allocating inventory (rationing) to high- and low-priority orders	(Q, R) inventory model with drop-shipping	Drop-shipping
Liu et al. (2007)	p, l	Linear $f(p, l)$ (1 product variety)	Decentralized (manufacturer as a leader and retailer as a follower) and Centralized	Optimizing and coordinating manufacturer-retailer operations	Stochastic single channel (M/M/1 queue as illustration)	Penalty cost for delays
Pekgün et al. (2008)	p, l	Linear $f(p, l)$ (1 product variety)	Decentralized (marketing and production departments) and Centralized	Lead time quotation and pricing to coordinate marketing and production departments	Stochastic single channel (M/M/1 queue)	No inventory
Khouja & Stylianou (2009)	S	Exogenous	Centralized (retailer)	Optimizing inventory policy with drop-shipping in case of shortage	(Q, R) inventory model with drop-shipping	Backorder or lost sale
Hua et al. (2010)	p, l	Linear $f(p, l)$ Substitution (2 varieties)	Centralized and Decentralized (manufacturer as a leader and retailer as a follower)	Optimizing and coordinating manufacturer-retailer operations	Deterministic dual (retail and direct) channel	No inventory
Zhao et al. (2012)	p, l	Linear $f(p, l)$ Substitution (2 varieties)	Centralized (service provider or MTO manufacturer)	Uniform DT quotation mode vs. differentiated quotation mode	Stochastic single or dual channels (one or two M/M/1 queues in parallel)	No inventory
Pekgün et al. (2017)	p, l	Linear $f(p, l)$ Substitution (2 Products)	Two competing companies (MTO manufacturers)	Competition on Lead time quotation and pricing	Stochastic with Two independent single channels (M/M/1 queues)	No inventory
Albana et al. (2018)	p, l	Linear $f(p, l)$ (1 product variety)	Centralized (MTO manufacturer)	Lead time quotation under lead time-dependent cost	Stochastic single channel (M/M/1 queue)	No inventory
Modak & Kelle (2019)	p, l, S	Linear $f(p, l)$ Substitution (2 varieties)	Centralized and Decentralized (manufacturer and retailer)	Optimizing and coordinating manufacturer-retailer operations	Dual online and traditional retail channels with random demand	Lost sales
Ma & Jemai (2019)	S	Exogenous	Centralized (retailer)	Inventory rationing to satisfy store and online demand	News-vendor model with drop-shipping as an additional fulfillment option	Penalty cost
Hammami et al. (2020a)	p, l	Linear $f(p, l)$ (1 product variety)	Centralized (MTO SC)	Lead time quotation and pricing in two-stage SC	Stochastic with two-stage SC (tandem queue M/M/1-M/M/1)	No inventory
Hammami et al. (2020b)	p, l	Linear $f(p, l)$ (1 product variety)	Centralized (MTO manufacturer)	Price and lead time quotation with customer	Stochastic single channel (M/M/1/K queue)	No inventory

		variety)		rejection		
Our study	p, l, S	Non-linear $f(p, l, S)$ Substitution (2 varieties)	Centralized (retailer)	Time-and price-based product differentiation and effect of stockout-based substitution	Stochastic dual channel with mix of DFS and drop-shipping (DFS + M/M/1)	Substitution + Lost sale

p =price, l = lead time, S = order size (inventory policy decisions), r = service level, DFS= delivery from stock, $f(\cdot)$ = function of.

3. Modeling framework

We consider a profit-maximizing retailer who uses a hybrid distribution with a mix of DFS and drop-shipping and, thus, offers two substitutable products differentiated in terms of DT and price to satisfy a random time- and price-sensitive demand. With drop-shipping, the retailer forwards the customer order to the supplier (wholesaler or manufacturer). The order is then processed by the supplier and delivered to the customer directly. In this case, the product is sold as a regular product. With traditional DFS, the product is immediately delivered to the customer from the retailer's stock (if the stock is available) and is sold as an express product. This SC structure has been illustrated in Figure 2.

Our initial demand model (i.e., without incorporating S-b-S) is linear with substitution. Customers arrive according to a Poisson process with arrival rates λ_1 and λ_2 for regular and express products, respectively. One unit of demand corresponds to one customer (order). The mean demand of each product is decreasing in its DT and price and increasing in the other product's DT and price. The regular product is offered at a standard price p_1 and a guaranteed DT l_1 . The express product is offered at a higher price p_2 and a shorter DT l_2 . The demand rates are given in equations (1) and (2). Note that the total potential market size is $2A$. Parameters α_p and β_l are price-sensitivity and time-sensitivity, respectively, and α_2 and β_2 represent the sensitivity of switchover toward price difference and DT difference, respectively.

$$\lambda_1 = A - \alpha_p p_1 + \alpha_2 (p_2 - p_1) - \beta_l l_1 + \beta_2 (l_2 - l_1) \quad (1)$$

$$\lambda_2 = A - \alpha_p p_2 + \alpha_2 (p_1 - p_2) - \beta_l l_2 + \beta_2 (l_1 - l_2) \quad (2)$$

Thus, if DT l_1 increases by one unit, $(\beta_2 + \beta_l)$ units of demand will be lost from the drop-shipping channel, of which β_2 units will transfer to the DFS channel, and β_l units will be lost from the two channels. Note that the total initial demand, given by $\lambda_1 + \lambda_2 = 2A - \alpha_p (p_1 + p_2) - \beta_l (l_1 + l_2)$, decreases in price charged and DT quoted. For the regular product, the supplier processes each single order forwarded by the retailer and ships it directly to the customer. The drop-shipping channel is then modeled as an MTO M/M/1 queue where the service time is exponentially distributed with rate μ_1 . The M/M/1 queue has been widely used in DT quotation literature. The retailer targets a minimum service level r_1 for the regular product. The probability that a customer is served within the quoted DT l_1 must be greater than r_1 (i.e., $\Pr(w \leq l_1) \geq r_1$, where w is the expected waiting time for a customer). This prevents an unreliable DT quote. As the express product is immediately delivered from stock, our entire analysis, without loss of generality, considers that l_2 is normalized to be equal to zero. Thus, the DT of regular product also represents the DT difference between

express and regular products. DT l_1 is a decision variable, which is of course affected by the drop-shipper service time (capacity). The retailer controls l_1 with pricing and inventory decisions as these decisions impact the demand in the drop-shipping channel.

Each replenishment of retailer's inventory refills the stock to its target level S . The service time to refill the stock is exponentially distributed with service rate μ_2 . The service time does not depend on the replenishment lot size since the products are assumed to be always available at the supplier's site, which is a common assumption (Zhu 2015). Thus, the replenishment time corresponds to preparation and transportation activities. We let ψ_0 denote the probability of stockout in the retailer's warehouse. The probability of serving customers (i.e., $1 - \psi_0$) must be greater than a predetermined minimum service level r_2 . In case of stockout, the customer can either switch to the regular product (i.e., accepts a longer DT l_1 but with a lower price p_1) or leave the system (lost sale). We let $\phi \in [0,1]$ denote the S-b-S rate; it represents the percentage of customers initially interested in the express product but accepting to switch to the regular product in case of stockout. In particular, $\phi = 1$ means that all customers switch to the regular product, whereas $\phi = 0$ means that all customers leave the system. Note that a smaller ϕ can represent a more competitive market; the S-b-S typically increases when the retailer is the exclusive seller of the product and decreases if the product can be found elsewhere by customers.

We let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ denote the effective demand rate of regular and express products, respectively. $\bar{\lambda}_1$ is the sum of customers initially interested in the regular product (i.e., λ_1) and customers interested in the express product but deciding to switch to the regular product in case of stockout (i.e., $\phi\psi_0\lambda_2$). $\bar{\lambda}_2$ is the difference between the customers initially interested in the express product (i.e., λ_2) and the customers that are likely to find empty stock (i.e., $\psi_0\lambda_2$). This system is illustrated in Figure 3.

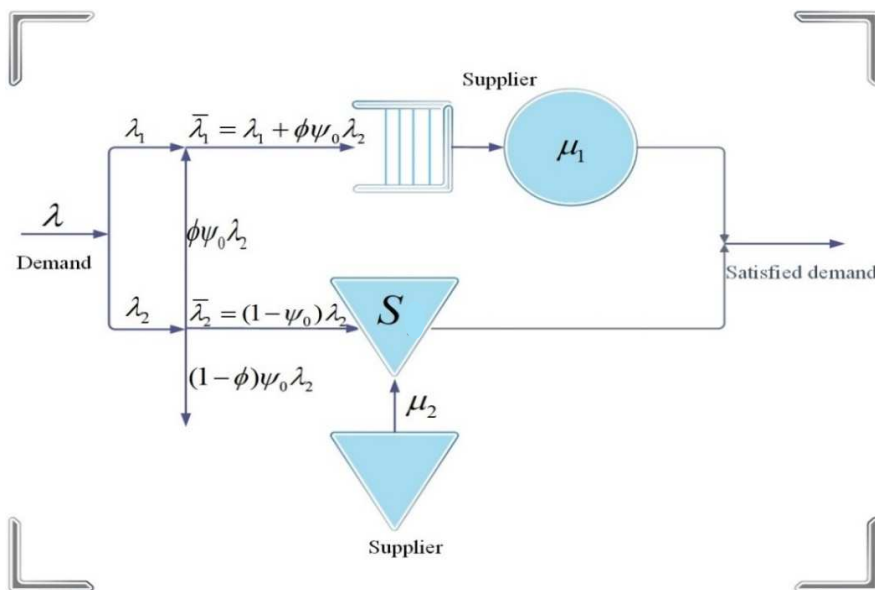


Figure 3. A hybrid delivery system with stockout-based substitution

To obtain a tractable model while capturing the essential features and key trade-offs, we make two approximations that we will discuss. The first approximation concerns the service constraint for the regular product. When the S-b-S is considered (i.e., for $\phi > 0$), the effective demand for the regular product is no longer exponentially distributed. In this case, we conducted extensive experiments to check whether it is still acceptable to use the service constraint formula of the M/M/1 queue (which assumes that demand is exponentially distributed). Thus, we simulated our hybrid system (shown in Figure 3) with ARENA software (version 14.0) while considering the following parameters: $\phi=0.7$, $\lambda_1=35$, $\lambda_2=40$, $\mu_1=40$, $\mu_2=20$, and $S=50$. We ran the system for the equivalent of 500,000 hours. The simulation's result revealed that the probability of serving customers on time is equal to 0.25495. Then, we assumed that the effective demand $\bar{\lambda}_1$ follows the exponential distribution and found that the probability of serving customers on time is equal to 0.25727. The gap between the probability obtained with simulation of the real system and the probability obtained with the exponential assumption is equal to 0.9% (less than 1%). To guarantee the robustness of this result, we repeated the same procedure 100 times while considering different values of parameters that were randomly generated as follows: $\phi \in [0.4, 1]$, $\lambda_1 \in [30, 40]$, $\lambda_2 \in [35, 45]$, $\mu_1 \in [35, 45]$, $\mu_2 \in [15, 25]$, and $S \in [45, 55]$. In all cases, we obtained the same result (i.e., the gap is smaller than 1%). Therefore, we conclude that the service constraint formula of the M/M/1 queue provides an acceptable approximation. Note that this constraint is exact for $\phi = 0$.

For the stockout probability, the general formula is very complex and leads to a non-tractable model. We therefore consider the stockout probability ψ_0 that we obtain if the reorder point (ROP) equals 0 (i.e., the buffer is refilled to stock level S when it is empty). For $\text{ROP}=0$, we obtain $\psi_0 = \left(1 + \frac{\mu_2 S}{\lambda_2}\right)^{-1}$ (see the proof in the appendix). Hence, in case of $\text{ROP}=0$, we study the system with the exact stockout probability. In case of $\text{ROP} > 0$, ψ_0 represents an upper bound of the exact stockout probability. The lower the ROP, the more precise this upper bound is. We explain at the end of this section that the considered stockout probability ψ_0 can be effectively used to study the case of $\text{ROP}>0$.

Thus, the effective demand rates are given by the following equations, where $\alpha_1 = \alpha_p + \alpha_2$ and $\beta_1 = \beta_l + \beta_2$ (l_2 is normalized to be equal to zero without loss of generality).

$$\bar{\lambda}_1 = \lambda_1 + \phi \psi_0 \lambda_2 = \lambda_1 + \phi \frac{(\lambda_2)^2}{\lambda_2 + \mu_2 S} = A - \alpha_1 p_1 + \alpha_2 p_2 - \beta_1 l_1 + \phi \frac{(A - \alpha_1 p_2 + \alpha_2 p_1 + \beta_2 l_1)^2}{A - \alpha_1 p_2 + \alpha_2 p_1 + \beta_2 l_1 + \mu_2 S} \quad (3)$$

$$\bar{\lambda}_2 = (1 - \psi_0) \lambda_2 = \frac{\mu_2 S \lambda_2}{\lambda_2 + \mu_2 S} = \frac{\mu_2 S (A - \alpha_1 p_2 + \alpha_2 p_1 + \beta_2 l_1)}{A - \alpha_1 p_2 + \alpha_2 p_1 + \beta_2 l_1 + \mu_2 S} \quad (4)$$

In the traditional shipping mode, the retailer purchases the product from the supplier for a wholesale price and then determines the retail price for consumers. We let c_2 denote the purchasing cost per unit of product (shipping cost included), which incurs the retailer's unit margin $m_2 = p_2 - c_2$ (this does not account for the inventory holding cost). In contrast, in the drop-shipping mode, the supplier determines the retail price and shares the revenue with the retailer. Consequently, the retailer loses the flexibility of setting market prices

(Khouja and Stylianou 2009, Tian et al. 2018). Hence, we consider that the regular product has a given fixed price and generates a fixed unit margin $m_1 = p_1 - c_1$ for the retailer.

In its general form, the problem is determining the DT differentiation (by quoting DT l_1), the price differentiation (by setting price p_2), and the stock level (by choosing S), with the objective of maximizing the total expected profit under service level constraints for both regular and express products. The general model is given below.

General model (M_{l_1, S, p_2})

$$\text{Max } \pi(l_1, S, p_2) = (p_1 - c_1)\bar{\lambda}_1 + (p_2 - c_2)\bar{\lambda}_2 - hS \quad (5)$$

Subject to

$$1 - e^{-(\mu_1 - \bar{\lambda}_1)l_1} \geq r_1 \quad (6)$$

$$\frac{\mu_2 S}{\lambda_2 + \mu_2 S} \geq r_2 \quad (7)$$

$$l_1 > 0, \quad S \geq 0, \quad p_2 > p_1, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \bar{\lambda}_1 < \mu_1.$$

Objective function (5) represents the total expected profit to be maximized (i.e., net revenue – inventory cost). Note that $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are given by equations (3) and (4), respectively. The exact formula of the expected stock level is $\sum_{i=0}^S i\psi_i = \frac{S+1}{2} \frac{1}{1 + \frac{\lambda_2}{\mu_2 S}}$ (where ψ_i represents the probability of having i items in stock when a customer arrives). However, this formula is not tractable since the model is already too complex. We make the following assumption that is widely used in theory and practice.

We consider that $\frac{\lambda_2}{\mu_2 S}$ is small (i.e., $\frac{\lambda_2}{\mu_2 S} \ll 1$). Note that $\frac{\lambda_2}{\mu_2 S} = \frac{1/\mu_2}{S/\lambda_2}$. Hence, the mean inventory replenishment time (i.e., $1/\mu_2$) is relatively small with comparison to the mean time required to consume the stock (i.e., S/λ_2). This condition is guaranteed by the service constraint for the express product. The case of a small $\frac{\lambda_2}{\mu_2 S}$ is widely adopted in the literature (see e.g., Rajagopalan 2002, Rafiei and Rabbani 2012) and reflects many practical situations such as when the retailer relies on a local supplier. A small value of $\frac{\lambda_2}{\mu_2 S}$ is consistent with the consideration of a low reorder point since the replenishment of the stock is relatively fast. When $\frac{\lambda_2}{\mu_2 S}$ is small, it is easy to show that the expected stock level is approximately equal to $\frac{S}{2}$.

The unit inventory cost is denoted by $2h$. Hence, the expected inventory cost is given by hS . Constraint (6) expresses the service level constraint for the regular product, i.e., $\Pr(w \leq l_1) \geq r_1$. We recall that w is the waiting time for a customer. Using the M/M/1 queue properties (as discussed earlier), we obtain $\Pr(w \leq$

$l_1) = 1 - e^{-(\mu_1 - \bar{\lambda}_1)l_1}$ (see Hammami et al. 2020a). The service level constraint for the express product, i.e., $1 - \psi_0 \geq r_2$, is given by constraint (7). We recall that $1 - \psi_0 = \frac{\mu_2 S}{\lambda_2 + \mu_2 S}$. The other constraints specify the variable domains. Note that the stability condition of the M/M/1 queue (i.e., $\bar{\lambda}_1 < \mu_1$) is automatically satisfied when constraint (6) is satisfied. Therefore, we can ignore the stability condition.

Finally, we recall that the stockout probability ψ_0 used in the model is exact for ROP = 0. In case of ROP > 0, ψ_0 is an upper bound of the exact stockout probability. Using this upper bound instead of the exact stockout probability does not change the main trade-offs of the model. It is also possible to minimize the impact of this approximation on the model's outcomes through an adequate adjustment of the values of some parameters. For instance, since ψ_0 is here an upper bound, we can run our model with a faster mean inventory replenishment lead time (i.e., a smaller $\frac{1}{\mu_2}$) to reduce the risk of stockout. In addition, in case of $\phi = 0$ (no S-b-S), we can decrease the minimum service level r_2 for the express product. In case of $\phi > 0$, we can also decrease the value of ϕ to reduce the number of switching customers in case of stockout. Furthermore, it is important to note that we can still use $\frac{S}{2}$ to model the expected stock level even when ROP > 0. We validate this result by simulation while considering different values of λ_2 , μ_2 and S as well as different values of ROP. More details about the simulation and its results are given in the appendix.

For convenience, we let l denote the DT of the regular product (instead of l_1) and p denote the price of the express product (instead of p_2) in the rest of the manuscript. In the following sections, we solve and analyze different variants of this model. In Section 4, we study the case where l is a decision variable while S and p are fixed. In Section 5, we study the case where l and S are decision variables while p is fixed. The general case, in which l , S , and p are decision variables, is studied in Section 6.

4. Delivery time quotation model with fixed stock and price

In this section, we determine the optimal DT l when S and p are fixed. To simplify the notation, we let $\lambda_1 = a_{1,p} - \beta_1 l$ and $\lambda_2 = a_{2,p} + \beta_2 l$, where $a_{1,p} = A - \alpha_1 p_1 + \alpha_2 p$ and $a_{2,p} = A + \alpha_2 p_1 - \alpha_1 p$. In addition, we let $z = \ln\left(\frac{1}{1-r_1}\right)$. The resulting model, denoted by (M_l) , is given below. Note that constraint (10) imposes lower and upper bounds on l to guarantee positive values for λ_1 and λ_2 (also note that $\frac{-a_{2,p}}{\beta_2}$ is not necessarily negative).

$$(M_l) \text{ Max } \pi(l) = m_1 \left(a_{1,p} - \beta_1 l + \phi \frac{(a_{2,p} + \beta_2 l)^2}{a_{2,p} + \beta_2 l + \mu_2 S} \right) + \frac{m_2 \mu_2 S}{a_{2,p} + \beta_2 l + \mu_2 S} (a_{2,p} + \beta_2 l) - hS$$

$$S.t. \quad \mu_1 - a_{1,p} + \beta_1 l - \phi \frac{(a_{2,p} + \beta_2 l)^2}{a_{2,p} + \beta_2 l + \mu_2 S} - \frac{z}{l} \geq 0 \quad (8)$$

$$\frac{\mu_2 S}{a_{2,p} + \beta_2 l + \mu_2 S} \geq r_2 \quad (9)$$

$$\max(0, \frac{-a_{2,p}}{\beta_2}) < l \leq \frac{a_{1,p}}{\beta_1} \quad (10)$$

As highlighted by many authors (e.g., [Ayanso et al. 2006](#), [Rabinovich et al. 2008](#)), the express product generates a higher margin than the regular product. Hence, we focus here on the case of $m_2 \geq m_1$. Since $\phi \leq 1$, we also have $m_2 \geq \phi m_1$. We first solve model (M_l) and then discuss the insights.

4.1. Model solving

Constraint (8) is equivalent to $f_\phi(l) \geq 0$ where $f_\phi(l) = \beta_2(\beta_1 - \phi\beta_2)l^3 + (\beta_1(a_{2,p} + \mu_2 S) - \beta_2(a_{1,p} - \mu_1 + 2\phi a_{2,p}))l^2 - ((a_{1,p} - \mu_1)(a_{2,p} + \mu_2 S) + \beta_2 z + \phi a_{2,p}^2)l - z(a_{2,p} + \mu_2 S)$. To solve model (M_l), we first provide a simpler formulation of constraint (8). All proofs are given in the appendix.

Lemma 1. For $\phi = 0$, the service constraint for the regular product (i.e., constraint (8)) is equivalent to $l \geq$

$$l_0, \text{ where } l_0 = \frac{a_{1,p} - \mu_1 + \sqrt{(a_{1,p} - \mu_1)^2 + 4\beta_1 z}}{2\beta_1}.$$

For $\phi > 0$, constraint (8) is equivalent to $l \geq l_\phi$, where l_ϕ is the unique root in $[l_0, \frac{a_{1,p}}{\beta_1}]$ of the cubic equation: $f_\phi(l) = 0$.

Lemma 1 provides a new lower bound on l . Furthermore, constraint (9) is equivalent to $l \leq \frac{(1-r_2)\mu_2 S - r_2 a_{2,p}}{r_2 \beta_2}$. Hence, we deduce that the feasible domain for model (M_l) is $[l_{min}, l_{max}]$, where $l_{min} = \max\{l_\phi, \frac{-a_{2,p}}{\beta_2}\}$ and $l_{max} = \min\{\frac{a_{1,p}}{\beta_1}, \frac{(1-r_2)\mu_2 S - r_2 a_{2,p}}{r_2 \beta_2}\}$. The optimal solution of model (M_l) is given below.

Lemma 2. The optimal DT of the regular product is:

$$l^* = \begin{cases} l_{th}, & \text{if } l_{min} \leq l_{th} \leq l_{max} \\ l_{min}, & \text{if } l_{th} < l_{min} \\ l_{max}, & \text{if } l_{th} > l_{max} \end{cases}$$

where $l_{th} = \frac{-a_{2,p} + \mu_2 S \left(\sqrt{\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)} - 1} \right)}{\beta_2}$, $l_{min} = \max\{l_\phi, \frac{-a_{2,p}}{\beta_2}\}$ and $l_{max} = \min\{\frac{a_{1,p}}{\beta_1}, \frac{(1-r_2)\mu_2 S - r_2 a_{2,p}}{r_2 \beta_2}\}$. For

$\phi = 0$ (i.e., in the case without S-b-S), $l_\phi = l_0 = \frac{a_{1,p} - \mu_1 + \sqrt{(a_{1,p} - \mu_1)^2 + 4\beta_1 z}}{2\beta_1}$.

For $\phi > 0$, l_ϕ is the unique solution in $[l_0, \frac{a_{1,p}}{\beta_1}]$ of the cubic equation $f_\phi(l) = 0$ (see Lemma 1).

4.2. Model analysis and insights

Cases $l^* = \frac{-a_{2,p}}{\beta_2}$ and $l^* = \frac{a_{1,p}}{\beta_1}$ correspond to situations where only one product is offered to the customers (the demand of the other product equals zero). Since we focus here on product differentiation, we exclude

these extreme cases from the analysis. Based on Lemma 2, we deduce a set of managerial implications in the following propositions. In Proposition 1, we analyze the time differentiation strategy.

Proposition 1. If $\frac{m_2}{m_1} \leq \frac{\beta_1}{\beta_2}$, then the retailer should adopt a minimum time differentiation strategy (i.e., offering the regular product with the shortest feasible DT).

If $\frac{\beta_1}{\beta_2} < \frac{m_2}{m_1} \leq \frac{1}{r_2^2} \left(\frac{\beta_1}{\beta_2} - \phi(1 - r_2^2) \right)$, then the optimal strategy depends on stock level S . The retailer should adopt a minimum time differentiation when S is smaller than a given threshold value, and a medium time differentiation (i.e., intermediate differentiation level between the minimum and the maximum) when S is larger than this threshold value.

If $\frac{m_2}{m_1} > \frac{1}{r_2^2} \left(\frac{\beta_1}{\beta_2} - \phi(1 - r_2^2) \right)$, then the optimal strategy is a maximum time differentiation (i.e., offering the regular product with the longest feasible DT).

The managerial guidelines provided in Proposition 1 are graphically illustrated in Figure 4.

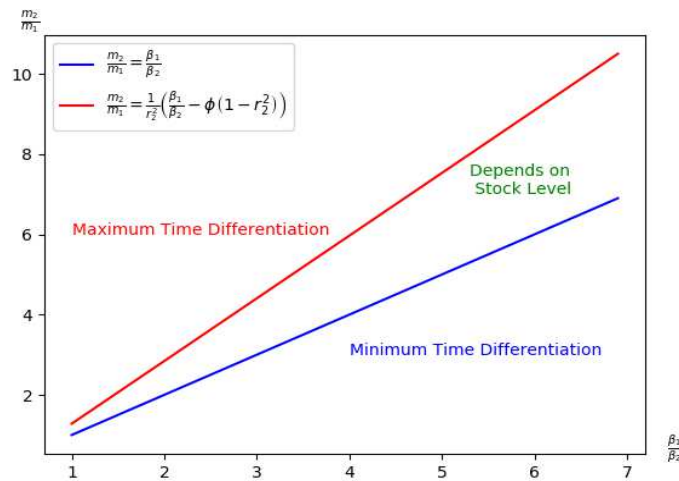


Figure 4. Time differentiation strategy when the characteristics of the express product are fixed

Note that $\frac{m_2}{m_1}$ represents the price differentiation between the express product and the regular product. Furthermore, $\frac{\beta_1}{\beta_2}$ represents the ratio of customers lost from the drop-shipping channel to customers transferring to the DFS channel due to a unitary increase in the DT of the regular product. When $\frac{m_2}{m_1} \leq \frac{\beta_1}{\beta_2}$, quoting a longer DT results in loss of revenue caused by a decrease in the regular demand that is greater than the additional revenue generated from the increase in the express demand. In this case, the retailer should quote the shortest feasible DT, which is given by $l^* = l_{min} = l_\phi$.

Then, when $\frac{\beta_1}{\beta_2} < \frac{m_2}{m_1} \leq \frac{1}{r_2^2} \left(\frac{\beta_1}{\beta_2} - \phi(1 - r_2^2) \right)$, the trade-off between the regular demand and the express demand is affected by the stock level. Compared to the first case, the express product becomes more

profitable. However, when S is relatively small, it is not in the retailer's interest to favor the express demand since the service level constraint for the express product cannot be satisfied. Thus, it is still optimal to adopt a minimum time differentiation. When the stock level increases, it is possible to capitalize more on the express demand, which leads to a greater time differentiation (i.e., $l^* = l_{th}$). Nevertheless, the price differentiation is here not large enough to justify maximizing the express demand, so there is no need for a maximum time differentiation. This situation is interesting as it shows that the optimal strategy is neither a minimum nor a maximum time differentiation but an intermediate strategy (with $l^* = l_{th}$) could be preferred.

Finally, when $\frac{m_2}{m_1} > \frac{1}{r_2^2} \left(\frac{\beta_1}{\beta_2} - \phi(1 - r_2^2) \right)$, the express product becomes profitable. Thus, the loss of revenue from the regular product, resulting from increasing the DT, is offset by the gain of revenue from the express product. It is then optimal to maximize the express demand (the only limitation is the service constraint), which leads to a maximum time differentiation (i.e., $l^* = l_{max}$).

To explain the practical implications of the above results, we first recall that a large ratio $\frac{\beta_1}{\beta_2}$ means a relatively big number of lost customers relative to the number of customers switching to the express product when the retailer quotes a longer DT. This characterizes, for instance, a highly competitive market. In such cases (i.e., for large values of $\frac{\beta_1}{\beta_2}$), we most likely have $\frac{m_2}{m_1} \leq \frac{\beta_1}{\beta_2}$ and, consequently, the retailers should offer the products with a minimum time differentiation. Alternatively, in markets with low competition, we most likely have $\frac{m_2}{m_1} \geq \frac{\beta_1}{\beta_2}$ and, consequently, the retailers should offer the products with a maximum time differentiation. The above analysis also reveals the following new result. While [Boyaci and Ray \(2003, 2006\)](#) and [Hua et al. \(2010\)](#) showed that the market characteristics govern the optimal product positioning and time differentiation strategies, we generalize this result and show that the inventory level also plays an important role. Indeed, while an extreme time differentiation strategy is proved to be optimal in most related works, we find that a medium time differentiation strategy may be optimal when the stock level is greater than a given threshold value. Proposition 2 focuses on the effect of stock level variation on time differentiation.

Proposition 2. A higher stock level S leads the retailer to quoting a shorter DT for the regular product as long as it is optimal to offer the two products with a minimum time differentiation (see the required conditions in Proposition 1), and to quoting a longer DT in the other cases.

It may be expected that a higher inventory would lead the retailer to favoring the express demand as this demand, served from the stock, generates a higher unit profit. Consequently, the retailer would quote a longer DT for the regular product to motivate the customers switching to the express product. However, in case of a minimum time differentiation, we demonstrate that the optimal DT decreases in S (see the proof in the appendix). The reason is that an increase in S reduces the number of switching customers in case of stockout, which implies a lower regular demand and, consequently, leads to a non-binding service constraint for the

regular product. Since we are in the case of a minimum time differentiation, it is optimal to quote the shortest feasible DT to the customers. Hence, the model reacts by quoting a shorter DT until it reaches the binding situation once again. Proposition 2 explains how the variation of retailer's stock level impacts the time differentiation strategy. This result cannot be obtained from studies that focus on MTO manufacturing systems (Boyaci and Ray 2003, 2006, Zhao et al. 2012, Pekgün et al. 2017). In addition, studies on inventory models with drop-shipping (such as Ayanso et al. 2006, Khouja and Stylianou 2009, Ma and Jemai 2019) did not address this question. We now study in Proposition 3 the effect of the S-b-S.

Proposition 3. An increase in S-b-S rate ϕ leads the retailer to offering the regular product with a longer DT as long as the retailer does not adopt a maximum time differentiation strategy. In case of a maximum time differentiation, the optimal DT does not depend on ϕ .

Recall that a greater ϕ implies a higher percentage of express demand transformed into regular demand in case of stockout. To understand the effect of S-b-S, three cases should be distinguished. In the case of a minimum time differentiation, the service constraint for the regular product is binding. Consequently, when ϕ increases, the system must quote a longer DT to decrease the regular demand and satisfy the service constraint. In the case of an intermediate time differentiation (i.e., when $l^* = l_{th}$), none of the service constraints are binding (neither for the regular product nor for the express product). This situation is less intuitive as the optimal strategy is a trade-off between regular and express demands and does not consist in maximizing one type of demand. In this case, the model reacts to an increase in ϕ by quoting a longer DT (similar to the first case) as this enables to offset the increase in the regular demand and to return to a more profitable trade-off. Finally, in case of a maximum time differentiation, the express product is much more profitable than the regular product, but the system cannot increase the express demand any more since the service constraint for the express product is already binding. Thus, an increase in ϕ does not lead to any reaction. We showed analytically how the S-b-S affects the time differentiation when S and p are fixed. In summary, a higher S-b-S leads to a higher time differentiation, except when the time differentiation is already at the maximum. This is a new result since our study is first to consider the S-b-S in time-based differentiation models.

5. Delivery time quotation and stock level determination model with fixed price

We now study the case where l and S are decision variables, but p is fixed. Recall that $\lambda_1 = a_{1,p} - \beta_1 l$ and $\lambda_2 = a_{2,p} + \beta_2 l$. The resulting model, denoted by $(M_{l,S})$, is given below.

$$(M_{l,S}) \text{Max } \pi(l, S) = m_1 \left(a_{1,p} - \beta_1 l + \phi \frac{(a_{2,p} + \beta_2 l)^2}{a_{2,p} + \beta_2 l + \mu_2 S} \right) + m_2 \mu_2 S \left(\frac{a_{2,p} + \beta_2 l}{a_{2,p} + \beta_2 l + \mu_2 S} \right) - hS$$

Subject to

$$\mu_1 - a_{1,p} + \beta_1 l - \phi \frac{(a_{2,p} + \beta_2 l)^2}{a_{2,p} + \beta_2 l + \mu_2 S} - \frac{z}{l} \geq 0 \quad (11)$$

$$\frac{\mu_2 S}{a_{2,p} + \beta_2 l + \mu_2 S} \geq r_2 \quad (12)$$

$l > 0, S \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0.$

Note that $\pi(l, S)$ is concave in S (for a given l) and reaches its maximum in $S_{th}(l) = \frac{\lambda_2}{\mu_2} \left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} - 1 \right)$. To obtain a relevant problem, we must have $\frac{\mu_2(m_2 - \phi m_1)}{h} \geq 1$. Service constraints (11) and (12) are, respectively, equivalent to $S \geq S_1(l) = \frac{\lambda_2}{\mu_2} \left(\frac{\phi \lambda_2}{\mu_1 - \lambda_1 - \frac{z}{l}} - 1 \right)$ and $S \geq S_2(l) = \frac{r_2}{1 - r_2} \frac{\lambda_2}{\mu_2}$. Hence, for a given feasible l , the optimal stock level is $S^*(l) = \max\{S_{th}(l), S_1(l), S_2(l)\}$. To make more progress, we need to distinguish two cases: $\phi = 0$ and $\phi > 0$.

5.1. Model $(M_{l,S})$ without stockout-based substitution ($\phi = 0$): Resolution and analytical insights

To solve model $(M_{l,S})$ for $\phi = 0$, we first provide a simpler equivalent formulation in Lemma 3.

Lemma 3. For $\phi = 0$, model $(M_{l,S})$ is equivalent to the following single-variable model.

$$\text{Max } \pi(l) = \begin{cases} m_1(a_{1,p} - \beta_1 l) + r_2 \left(m_2 - \frac{h}{(1 - r_2)\mu_2} \right) (a_{2,p} + \beta_2 l) & \text{if } m_2 \leq \frac{h}{(1 - r_2)^2 \mu_2} \\ m_1(a_{1,p} - \beta_1 l) + \left(\sqrt{m_2} - \sqrt{\frac{h}{\mu_2}} \right)^2 (a_{2,p} + \beta_2 l) & \text{if } m_2 > \frac{h}{(1 - r_2)^2 \mu_2} \end{cases}$$

Subject to $\max\left\{l_0, \frac{-a_{2,p}}{\beta_2}\right\} \leq l \leq \frac{a_{1,p}}{\beta_1}$, where $l_0 = \frac{a_{1,p} - \mu_1 + \sqrt{(a_{1,p} - \mu_1)^2 + 4\beta_1 z}}{2\beta_1}$.

Note that $\pi(l)$ is linear in l . This result is not intuitive; it implies that the optimal DT is either the shortest feasible DT (i.e., $\max\left\{l_0, \frac{-a_{2,p}}{\beta_2}\right\}$) or the longest DT (i.e., $\frac{a_{1,p}}{\beta_1}$). When it is optimal to quote the shortest DT, we assume that $l_0 \geq \frac{-a_{2,p}}{\beta_2}$ since, otherwise, the retailer offers only the regular product, which is an extreme case that we do not study. Lemma 4 characterizes the optimal retailer's strategy under different conditions. These optimal strategies are then recapitulated in Table 2.

Lemma 4. For $\phi = 0$, the optimal solution of model $(M_{l,S})$ is given as follows.

Case of $m_2 \leq \frac{h}{(1 - r_2)^2 \mu_2}$

$$\begin{aligned}
- \text{ If } m_2 \leq \frac{m_1\beta_1}{r_2\beta_2} + \frac{h}{(1-r_2)\mu_2}, \text{ then } & \begin{cases} l^* = l_0 = \frac{a_{1,p}-\mu_1 + \sqrt{(a_{1,p}-\mu_1)^2 + 4\beta_1 z}}{2\beta_1} \\ S^* = \left(\frac{r_2}{1-r_2}\right) \frac{2\beta_1 a_{2,p} + \beta_2 (a_{1,p}-\mu_1 + \sqrt{(a_{1,p}-\mu_1)^2 + 4\beta_1 z})}{2\beta_1 \mu_2} \end{cases} \\
- \text{ If } m_2 > \frac{m_1\beta_1}{r_2\beta_2} + \frac{h}{(1-r_2)\mu_2}, \text{ then } & \begin{cases} l^* = \frac{a_{1,p}}{\beta_1} \\ S^* = \left(\frac{r_2}{1-r_2}\right) \frac{\beta_1 a_{2,p} + \beta_2 a_{1,p}}{\beta_1 \mu_2} \end{cases}
\end{aligned}$$

Case of $m_2 > \frac{h}{(1-r_2)^2\mu_2}$

$$\begin{aligned}
- \text{ If } m_2 \leq \left(\sqrt{\frac{m_1\beta_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}}\right)^2, \text{ then } & \begin{cases} l^* = l_0 = \frac{a_{1,p}-\mu_1 + \sqrt{(a_{1,p}-\mu_1)^2 + 4\beta_1 z}}{2\beta_1} \\ S^* = \left(\sqrt{\frac{m_2\mu_2}{h}} - 1\right) \frac{2\beta_1 a_{2,p} + \beta_2 (a_{1,p}-\mu_1 + \sqrt{(a_{1,p}-\mu_1)^2 + 4\beta_1 z})}{2\beta_1 \mu_2} \end{cases} \\
- \text{ If } m_2 > \left(\sqrt{\frac{m_1\beta_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}}\right)^2, \text{ then } & \begin{cases} l^* = \frac{a_{1,p}}{\beta_1} \\ S^* = \left(\sqrt{\frac{m_2\mu_2}{h}} - 1\right) \frac{\beta_1 a_{2,p} + \beta_2 a_{1,p}}{\beta_1 \mu_2} \end{cases}
\end{aligned}$$

Table 2. Optimal retailer's strategy under the setting of model $(M_{l,S})$ with $\phi = 0$

$m_2 \leq \frac{h}{(1-r_2)^2\mu_2}$		$m_2 > \frac{h}{(1-r_2)^2\mu_2}$	
$m_2 \leq \frac{m_1\beta_1}{r_2\beta_2} + \frac{h}{(1-r_2)\mu_2}$	$m_2 > \frac{m_1\beta_1}{r_2\beta_2} + \frac{h}{(1-r_2)\mu_2}$	$m_2 \leq \left(\sqrt{\frac{m_1\beta_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}}\right)^2$	$m_2 > \left(\sqrt{\frac{m_1\beta_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}}\right)^2$
Minimum time differentiation	Only the express product is offered (DFS channel)	Minimum time differentiation	Only the express product is offered (DFS channel)

Based on the result of Lemma 4, we derive insights into the optimal strategy of the retailer in the following two propositions. In Proposition 4, we investigate the effect of prices on the time differentiation.

Proposition 4. There exists a threshold limit on the express product's price below which it is optimal to offer both products with a minimum time differentiation, and above which only the express product should be offered (i.e., the retailer must rely only on the DFS channel). This threshold price depends on the regular product's margin as follows:

$$- \text{ If } m_1 \leq \frac{h\beta_2}{\mu_2\beta_1} \left(\frac{r_2}{1-r_2}\right)^2, \text{ then the threshold price is } \frac{\beta_1 m_1}{r_2\beta_2} + \frac{h}{(1-r_2)\mu_2} + c_2,$$

- If $m_1 > \frac{h\beta_2}{\mu_2\beta_1} \left(\frac{r_2}{1-r_2}\right)^2$, then the threshold price is $\left(\sqrt{\frac{\beta_1 m_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}}\right)^2 + c_2$.

Observing Table 2, we expect that an increase in m_2 would have a non-monotonous effect on the time differentiation strategy. However, whatever the values of the parameters, we demonstrate that the intermediate intervals (second and third columns of Table 2) cannot be feasible simultaneously (see the proof of Proposition 4 in the appendix). Therefore, when m_2 increases, the model reacts only by moving from a minimum time differentiation to giving up the drop-shipping channel. Thus, unlike model (M_l) (in which the stock level was fixed), it is never optimal here to have an intermediate time differentiation strategy. There are only two potential optimal strategies: offering both products with a minimum time differentiation or offering only the express product. In Proposition 5, we investigate the case where both products are offered to the customers.

Proposition 5. If both products are offered to the customers, then the higher the express product's price is, the longer the DT of the regular product and the higher the stock level are.

Proposition 5 investigates the impact of the price differentiation and indicates, as expected, that the optimal DT and the optimal stock are increasing in the express product's price. The reason is that a longer DT for the regular product and a higher stock favor the express demand. Thus, the higher the price differentiation is, the higher the time differentiation becomes, which confirms the findings of [Modak and Kelle \(2019\)](#) and [Boyaci and Ray \(2003, 2006\)](#). We complement these findings by showing that a higher price differentiation leads the retailer to hold more stock. Finally, note that the results discussed in this section are relative to the case without S-b-S and that the consideration of S-b-S will likely impact such trade-offs. This is the focus of the next section.

5.2 Model ($M_{l,S}$) with stockout-based substitution ($\phi > 0$): Analysis and numerical insights

When $\phi > 0$, it is not possible to solve model ($M_{l,S}$) analytically. The results are then based on numerical analyses. We first provide a simpler formulation of the model and then use it to conduct our analysis.

Lemma 5. For $\phi > 0$, Model ($M_{l,S}$) is equivalent to the following single-variable model:

Max $\pi(l)$

$$= \begin{cases} \pi_1(l) = \left(m_1 - \frac{m_2}{\phi}\right)\mu_1 + m_2 a_{2,p} + \frac{m_2 a_{1,p}}{\phi} + \frac{h a_{2,p}}{\mu_2} - \left(m_1 - \frac{m_2}{\phi}\right)\frac{z}{l} + \left(m_2 \beta_2 - \frac{m_2 \beta_1}{\phi} + \frac{h \beta_2}{\mu_2}\right)l \\ \quad - \left(\frac{h\phi}{\mu_2}\right) \frac{(a_{2,p} + \beta_2 l)^2}{\mu_1 - a_{1,p} + \beta_1 l - \frac{z}{l}} \quad \text{if } l_0 \leq l \leq l_\theta \\ \pi_2(l) = m_1(a_{1,p} - \beta_1 l) + \left(m_2 + \frac{h}{\mu_2} - \frac{h}{\mu_2}\theta - \frac{m_2 - \phi m_1}{\theta}\right)(a_{2,p} + \beta_2 l) \quad \text{if } l_\theta \leq l \leq \frac{a_{1,p}}{\beta_1} \end{cases}$$

$$\text{where } \theta = \max\left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}}, \frac{1}{1-r_2}\right) \text{ and } l_\theta = \frac{(\theta(a_{1,p} - \mu_1) + \phi a_{2,p}) + \sqrt{(\theta(a_{1,p} - \mu_1) + \phi a_{2,p})^2 + 4\theta(\theta\beta_1 - \phi\beta_2)z}}{2(\theta\beta_1 - \phi\beta_2)}.$$

Note that when $\phi = 0$, we have $l_\theta = l_0$, implying that $\pi_1(l)$ is no longer defined and that $\pi(l) = \pi_2(l)$ over $[l_0, \frac{a_{1,p}}{\beta_1}]$. Thus, we obtain the same linear profit function given in the previous section (Lemma 3). When $\phi > 0$, we have two different behaviors of profit in function of l represented by the two functions $\pi_1(l)$ and $\pi_2(l)$ defined, respectively, over $[l_0, l_\theta]$, and $[l_\theta, \frac{a_{1,p}}{\beta_1}]$. $\pi_1(l)$ is not linear while $\pi_2(l)$ is linear. Therefore, the optimal DT is either the DT that maximizes $\pi_1(l)$ over $[l_0, l_\theta]$, or l_θ or $\frac{a_{1,p}}{\beta_1}$. Given the complexity of $\pi_1(l)$, it is not possible to obtain a closed-form expression of the optimal solution. Hence, the remainder of our analysis will be based on numerical experiments. We consider three market structures as in [Boyaci and Ray \(2003\)](#). The first market is characterized by $\frac{\beta_2}{\beta_l} > \frac{\alpha_2}{\alpha_p}$, i.e., the switchovers are more governed by time difference than price difference, which is referred to as a Switchovers Time-Driven (STD) market. The second market is characterized by $\frac{\beta_2}{\beta_l} < \frac{\alpha_2}{\alpha_p}$, i.e., the switchovers are more governed by price difference than time difference, which is referred to as a Switchovers Price-Driven SPD market. When $\frac{\beta_2}{\beta_l} = \frac{\alpha_2}{\alpha_p}$, the market is neutral (neither STD nor SPD). Similar to [Boyaci and Ray \(2003\)](#), we consider the following parameters for each market structure. For the neutral market, $\alpha_p = 50$, $\alpha_2 = 10$, $\beta_l = 50$, and $\beta_2 = 10$. For the STD market, $\alpha_p = 50$, $\alpha_2 = 10$, $\beta_l = 25$, and $\beta_2 = 25$. For the SPD market, $\alpha_p = 25$, $\alpha_2 = 25$, $\beta_l = 50$, and $\beta_2 = 10$. In our numerical experiments, the values of reference for most parameters are also based on the numerical example used in [Boyaci and Ray \(2003\)](#). For the unit inventory holding cost, which is not considered in [Boyaci and Ray \(2003\)](#), we make it consistent with the price of the product. For the S-b-S rate, we cover all significant values (from 0.2 to 1 with a step of 0.2). Then, for each type of market, we generate 168,000 instances by varying the values of the parameters, as shown in Table 3.

Table 3. Test cases

Parameter	Generated values
A	800, 900, 1000, 1100, 1200
μ_1	150
μ_2	200, 250, 300, 350, 400
c_1	3
c_2	3.5, 4, 4.5, 5
h	0.5, 1, 1.5, 2
p_1	6, 7, 8, 9, 10, 11, 12
p_2	For each pair (p_1, c_2) , we vary p_2 from $(p_1 - c_1 + c_2)$ to 18 with a step of 1

ϕ	0.2, 0.4, 0.6, 0.8, 1
$r_1 = r_2$	0.98

The analysis of the results leads to the following observations.

Observation 1. If $\left(-\beta_1 m_1 + \beta_2 \left(m_2 + \frac{h}{\mu_2} - \frac{h}{\mu_2} \theta - \frac{m_2 - \phi m_1}{\theta}\right)\right) > 0$ (i.e., if $\pi_2(l)$ is increasing in l), then it is more profitable to offer only the express product (i.e., $l^* = \frac{a_1 p}{\beta_1}$) and to use the drop-shipping channel as a backup solution in case of stockout.

This result has been observed in all instances in which $\pi_2(l)$ is increasing (39,875 instances in total). To illustrate, we consider the example of Figure 5 ($A=1000$, $p_1 = 10$, $c_1 = 3$, $\mu_1 = 150$, $p_2 = 18$, $c_2 = 4.5$, $\mu_2 = 400$, $h = 1$, $r_1 = r_2 = 0.98$, and an STD market with $\phi = 0.80$). We observe that the profit is increasing in l and, consequently, the optimal solution is offering the longest DT for the regular product. This means that the initial regular demand is equal to zero and that only the express product is to be offered.

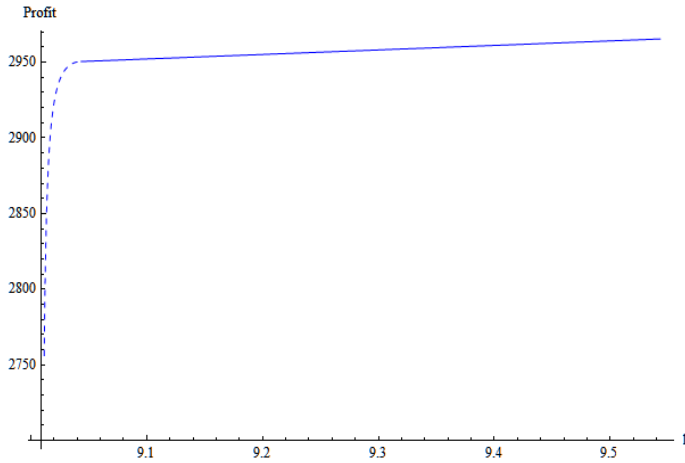


Figure 5. Profit in function of DT for increasing $\pi_2(l)$

Observation 2. If $\left(-\beta_1 m_1 + \beta_2 \left(m_2 + \frac{h}{\mu_2} - \frac{h}{\mu_2} \theta - \frac{m_2 - \phi m_1}{\theta}\right)\right) < 0$ (i.e., if $\pi_2(l)$ is decreasing in l), then it is optimal to offer both products. In this case, quoting the DT l_θ is a near-optimal solution.

The case where $\pi_2(l)$ is decreasing in l is the most observed case. For the neutral market, we obtained 56,705 feasible instances. In all of them, we have $\left(-\beta_1 m_1 + \beta_2 \left(m_2 + \frac{h}{\mu_2} - \frac{h}{\mu_2} \theta - \frac{m_2 - \phi m_1}{\theta}\right)\right) < 0$, implying that the linear function $\pi_2(l)$ is decreasing in l . For the SPD market, we obtained 32,031 feasible instances, and $\pi_2(l)$ is also decreasing in all of them. Finally, for the STD market, $\pi_2(l)$ is decreasing in 24,904 instances and increasing in 39,875 instances.

For each instance with decreasing $\pi_2(l)$ (113,640 instances in total), we calculated the optimal DT and deduced the profit gap resulting from quoting l_θ (which is the intersection point between $\pi_1(l)$ and $\pi_2(l)$) instead of the real optimal DT. The gap is calculated as follows: $\frac{Profit(optimal\ DT) - Profit(l_\theta)}{Profit(optimal\ DT)} \times 100$. For all market types, we obtained a minimal mean gap (0.007% for the neutral market, 0.0002% for the STD market, and 0.012% for the SPD market). This shows that l_θ is a near-optimal DT. To illustrate, we consider the following example: $A=1000$, $p_1 = 10$, $c_1 = 3$, $\mu_1 = 150$, $p_2 = 14$, $c_2 = 4.5$, $\mu_2 = 400$, $h = 1$, $r_1 = r_2 = 0.98$, with a neutral market structure and different values of ϕ . The profiles of the profit functions are shown in Figure 6 (note that $\pi_1(l)$ and $\pi_2(l)$ are, respectively, represented by dashed and continuous lines over $[l_0, l_\theta]$ and $[l_\theta, \frac{a_1 p}{\beta_1}]$, and that l_θ is the intersection point). For $\phi = 0.2$, $\pi_1(l)$ is increasing over $[l_0, l_\theta]$ and, consequently, $l^* = l_\theta$ (since $\pi_2(l)$ is decreasing). For the other values of ϕ , $\pi_1(l)$ first increases and then slightly decreases before l_θ . We can see that the curve is flat between the optimal DT and l_θ , which means that l_θ is near-optimal.

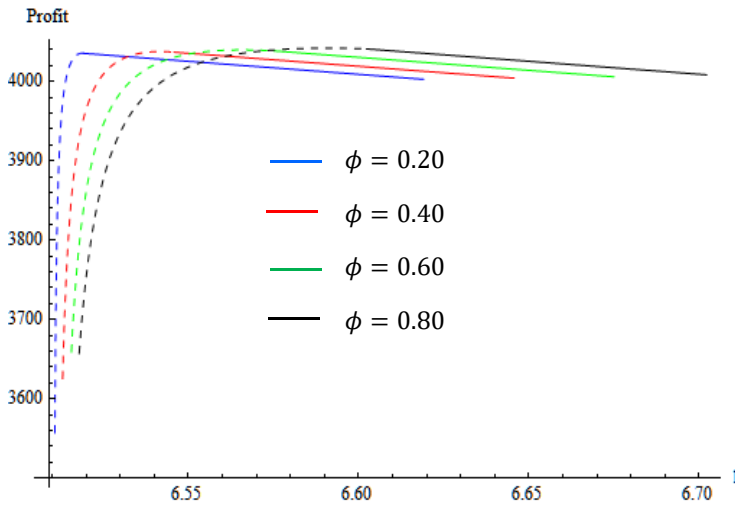


Figure 6. Profit in function of DT when $\pi_2(l)$ is decreasing

Observation 3. The S-b-S leads to a higher time differentiation (i.e., offering the regular product with a longer DT), and ignoring the S-b-S (by quoting the optimal DT obtained for $\phi = 0$, namely l_0) leads to a considerable loss.

Since $l_\theta > l_0$, it is clear that the consideration of the S-b-S leads to a retailer quoting a longer DT. In addition, similar to the results obtained from model (M_l), we observe in Figure 6 that the higher the value of ϕ is, the longer the optimal DT for the regular product is. Note that $\pi_1(l)$ increases rapidly at the neighborhood of l_0 (for all values of ϕ). This means that ignoring the S-b-S, and, consequently, quoting the optimal DT obtained for $\phi = 0$ (i.e., l_0) can lead to a substantial loss. We quantified this loss by calculating

the gap $\frac{Profit(l_\theta) - Profit(l_0)}{Profit(l_0)} \times 100$. We found a mean gap of 39.36% for the neutral market, 57.18% for the SPD market, and 45.28% for the STD market.

Our analysis sheds light on an important new finding that has not been revealed in the extant literature. We show that the existence of stockout-based switching customers must lead retailers to increase the time differentiation, otherwise their expected profit will significantly decrease. In practice, this result means that retailers should quantify the S-b-S rate and select slower drop-shippers when the S-b-S increases. This is particularly relevant to leading retailers or retailers who are the exclusive sellers of some products. As the S-b-S rate is usually high for these retailers, it may be useless to offer fast drop-shipping channels. Note finally that our model generalizes many frameworks that consider stockout in the literature (e.g., [Liu et al. 2007](#), [Khouja and Stylianou 2009](#), [Ma and Jemai 2019](#), [Modak & Kelle 2019](#)) as the optimal strategy can be either to offer two differentiated products, or to use the drop-shipping only as a backup solution in case of stockout. In this latter case, the whole system can be viewed as a traditional shipping system with a combination of lost sales (customers who leave in case of stockout) and back-orders (customers who switch to drop-shipping in case of stockout).

6. General model

In this section, we study the general model $(M_{l,S,p})$ where l, S , and p are decision variables and the S-b-S is considered. Given the complexity of the model, we first provide a simpler quasi-equivalent formulation using the results of the previous section. Then, we rely on the new formulation to conduct experiments and derive managerial implications. All the results of this section are based on numerical experiments.

6.1. Model reformulation

To simplify the analysis, we focus on the more general case where both regular and express products are offered to the customers. Therefore, we consider the values of p such that $\left(-\beta_1 m_1 + \beta_2 \left(p - c_2 + \frac{h}{\mu_2} - \frac{h}{\mu_2} \theta - \frac{p - c_2 - \phi m_1}{\theta}\right)\right) < 0$ (see Observations 1 and 2). Under this condition, and for a given price p , we have shown in the previous section that the model can be written in function of the single variable l and that $l = l_\theta$ is a near-optimal solution (for $\phi = 0$, $l_\theta = l_0$ is the exact optimal solution). Hence, for a given price p , we replace l with l_θ . Then, we show by standard (but long) calculus, that model $(M_{l,S,p})$ can be written in function of the single variable p . We denote this new model by (M) and provide it below. We recall that the express product generates a higher margin than the regular product (i.e., $p - c_2 \geq m_1$). In addition, we have explained in the case of fixed price that we must have $\frac{\mu_2(m_2 - \phi m_1)}{h} \geq 1$ since, otherwise, it is not possible to

generate a positive profit. This constraint is equivalent to $p \geq \frac{h}{\mu_2} + \phi m_1 + c_2$. Hence, we finally have $p \geq$

$p_{min} = \max \left\{ m_1 + c_2, \frac{h}{\mu_2} + \phi m_1 + c_2 \right\}$. This constraint is added to the general model.

$$(M) \text{ Max}_{p \geq p_{min}} \pi(p) = \begin{cases} \pi_3(p) = m_1(a_1 + \alpha_2 p - \beta_1 l_\theta(p)) + \left(r_2(p - c_2) - \frac{r_2 h}{(1-r_2)\mu_2} + \phi(1-r_2)m_1 \right) (a_2 - \alpha_1 p + \beta_2 l_\theta(p)) \\ \text{if } p \leq \min \left\{ \frac{m_1 \beta_1}{r_2 \beta_2} + \frac{h}{(1-r_2)\mu_2} - \frac{\phi(1-r_2)m_1}{r_2} + c_2, \frac{h}{(1-r_2)^2 \mu_2} + \phi m_1 + c_2 \right\} \\ \\ \pi_4(p) = m_1(a_1 + \alpha_2 p - \beta_1 l_\theta(p)) + \left(\left(\sqrt{p - c_2 - \phi m_1} - \sqrt{\frac{h}{\mu_2}} \right)^2 + \phi m_1 \right) (a_2 - \alpha_1 p + \beta_2 l_\theta(p)) \\ \text{if } \frac{h}{(1-r_2)^2 \mu_2} + \phi m_1 + c_2 \leq p \leq \left(\sqrt{\frac{(\beta_1 - \phi \beta_2)m_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}} \right)^2 + \phi m_1 + c_2 \end{cases}$$

where $p_{min} = \max \left\{ m_1 + c_2, \frac{h}{\mu_2} + \phi m_1 + c_2 \right\}$,

$$l_\theta(p) = \frac{(\theta(a_1 + \alpha_2 p - \mu_1) + \phi(a_2 - \alpha_1 p)) + \sqrt{(\theta(a_1 + \alpha_2 p - \mu_1) + \phi(a_2 - \alpha_1 p))^2 + 4\theta(\theta\beta_1 - \phi\beta_2)z}}{2(\theta\beta_1 - \phi\beta_2)},$$

$$\theta = \frac{1}{1-r_2} \text{ if } p \leq \frac{h}{(1-r_2)^2 \mu_2} + \phi m_1 + c_2 \text{ (i.e., for } \pi_3(p)), \text{ and } \theta = \sqrt{\frac{\mu_2(p - c_2 - \phi m_1)}{h}} \text{ otherwise (i.e., for } \pi_4(p)).$$

Before proceeding further, we shall verify that model (M) provides a good approximation of the original model ($M_{l,S,p}$). We generate instances while varying the values of the different parameters according to the rules described in Table 3. For the S-b-S rate ϕ , we consider the following values: 0, 0.2, 0.4, 0.6, 0.8, and 1. We obtain 7401 feasible instances with $\phi = 0$ and 36363 feasible instances with $\phi > 0$. For each instance, we calculate the gap between the optimal profit obtained with model (M) and that of the original model ($M_{l,S,p}$). To solve model ($M_{l,S,p}$), we used the algorithm SLSQP (sequential least squares programming) of the SciPy module. This is a recommended procedure for nonlinear optimization problems with nonlinear constraints. The modified model (M) can be easily solved to optimality with any computational software. The gap between both models is calculated as $\frac{(\pi_{original\ model} - \pi_{modified\ model}) * 100}{\pi_{original\ model}}$. Table 4 shows the results. Notice that we consider the value of confidence 95%.

Table 4. Comparison between the original model and the modified model

	$\phi = 0$				$\phi > 0$			
	Test Cases: 2,800				Test Cases: 14,000			
Market	Nb. Relevant Instances	Mean Gap	Standard Deviation	Confidence Interval	Nb. Relevant Instances	Mean Gap	Standard Deviation	Confidence Interval

Neutral	2449	0.0002	0.0003	(0.00022, 0.00024)	11955	0.0087	0.0294	(0.0082, 0.0092)
SPD	2798	0.00014	0.0001	(0.00013, 0.00015)	13917	0.0102	0.0494	(0.0094, 0.0110)
STD	2154	0.95	3.0392	(0.8216, 1.0783)	10491	0.8560	2.7181	(0.8036, 0.9076)

Table 4 shows that the mean profit gap is close to zero, particularly for the neutral and SPD markets. This confirms that the modified model (M) is quasi-equivalent to the original model ($M_{l,S,p}$). This small gap between the optimal profit of model (M) and that of model ($M_{l,S,p}$) is due to the use of the DT l_θ in the modified model instead of the optimal DT (we have shown that l_θ is a near-optimal solution for a given price p).

6. 2. Numerical analysis

The remainder of our analysis will be based on model (M). Our main objective is to investigate whether the effect of the S-b-S observed in the previous models still holds here. We consider the basic example: $A=1000$, $p_1 = 8$, $c_1 = 3$, $\mu_1 = 150$, $c_2 = 4.5$, $\mu_2 = 400$, $h = 1$, and $r_1 = r_2 = 0.98$. We vary ϕ and present the results in the following figures.

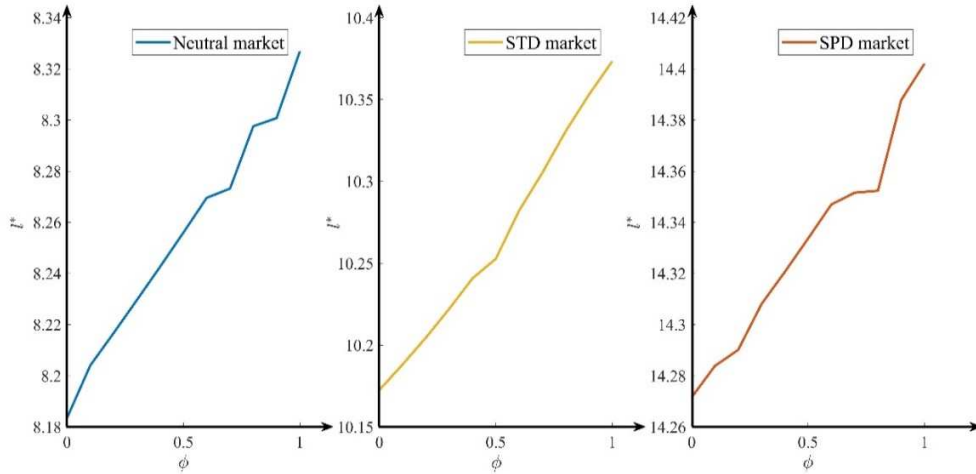


Figure 7. Effect of stockout-based substitution on DT quotation

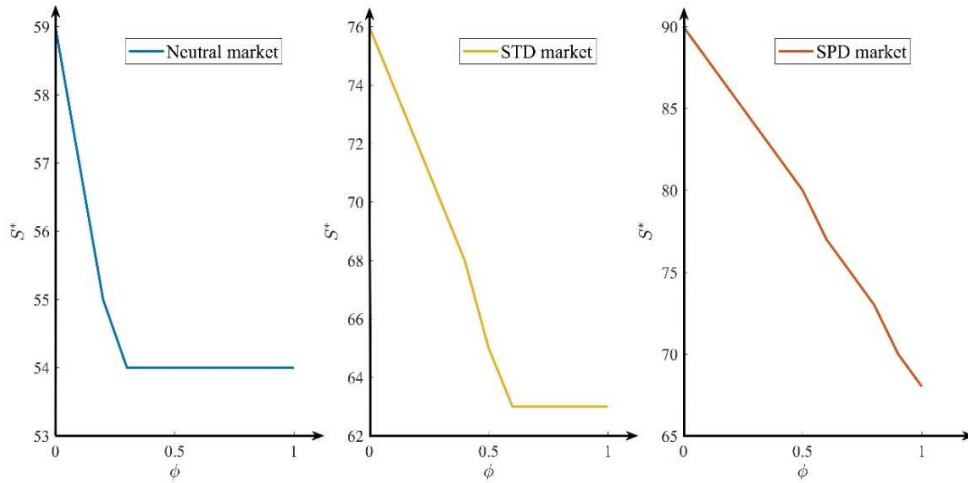


Figure 8. Effect of stockout-based substitution on inventory

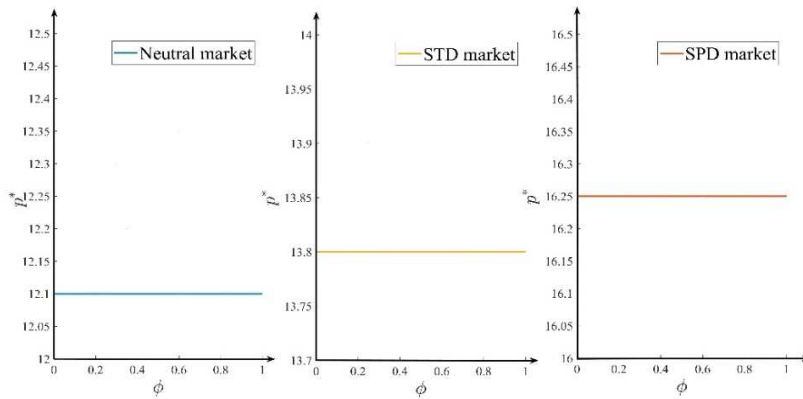


Figure 9. Effect of stockout-based substitution on price

The analysis of the numerical results leads to the following main finding.

Observation 4. In the general model, a higher S-b-S rate would normally lead to a greater time differentiation and to holding less stock. However, it would not impact the price differentiation.

When ϕ increases, different trade-offs between the price, DT, and stock govern the system. Nevertheless, we observe for all types of markets that an increase in the S-b-S rate does not affect the optimal price p^* . This result is not intuitive and implies that the price quoted for the express product should be the same whatever the extent of S-b-S. In particular, the optimal price is unchanged with or without S-b-S. To confirm this result, we extend our experiments to test, for each type of market, all the instances that are generated according to Table 3. We proceed as follows. For each instance that has $\phi > 0$, we solve the modified model (M) and obtain the optimal price $p_{(\phi>0)}^*$. Then, for this same instance, we set $\phi = 0$ and solve model (M) again to obtain the new

optimal price that corresponds to $\phi = 0$. We denote this new price by $p_{(\phi=0)}^*$. Finally, we calculate the price difference $p_{(\phi>0)}^* - p_{(\phi=0)}^*$. We calculate this price difference for each instance generated in Table 3. The results are provided in Table 5. We see that the mean price difference (for all instances) is very close to zero, which confirms that the optimal price is not sensitive to the S-b-S. To confirm these results, we extend the experiments to check whether our findings still hold for the original model with fixed S . We solve the resulting model numerically. Then, we vary ϕ and observe its impact on the price and DT. The results show that the DT differentiation increases, and the prices are constant for increasing values of ϕ . Hence, the findings of Observation 4 are still valid (we do not present these experiments here due to space constraints).

Table 5. Comparison of optimal prices with $\phi > 0$ and $\phi = 0$

Market	Mean price difference	Standard deviation	Confidence interval
Neutral	0.0046	0.0066	(0.0044, 0.0049)
SPD	0.0080	0.0083	(0.0077, 0.0083)
STD	0.0769	0.2326	(0.0683, 0.0856)

The above results were illustrated in Figures 7, 8 and 9. We see that an increase in the S-b-S leads to a greater time differentiation and to holding less stock but does not impact the price. When ϕ increases, it is preferable to capitalize on the express demand since the loss of the regular demand can be offset by the additional number of customers switching from the express product to the regular product in case of stockout. Thus, as it is not profitable to reduce the price of the express product (we have just shown that this price does not change), the system reacts by quoting a longer DT for the regular product. In addition, when ϕ increases, the stockout becomes less penalizing for the retailer as there is a smaller proportion of customers leaving the system. Thus, the system favors reducing the inventory cost over satisfying more express demand. This explains why the stock decreases when ϕ increases. Nevertheless, beyond a threshold value, it is no longer possible to reduce the stock since the system must satisfy the service constraint. Therefore, the stock remains constant, as we observe in Figure 8.

Finally, we investigate the effect of service rates μ_1 and μ_2 on the retailer's decisions. We represent the results for the neutral market in the following figures. We have obtained similar results for the other types of markets (we do not provide these results due to space constraints).

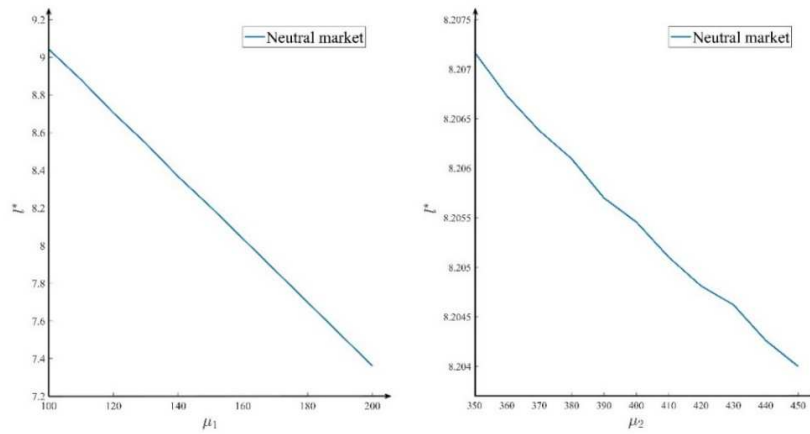


Figure 10. Effect of capacities on DT quotation

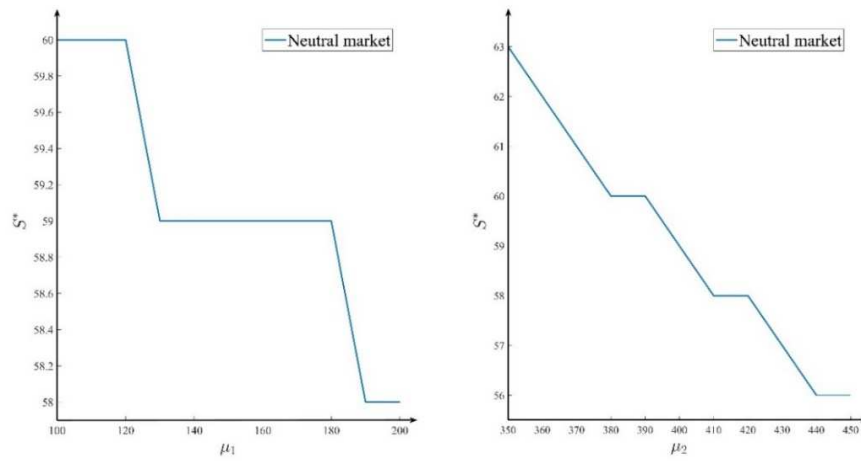


Figure 11. Effect of capacities on inventory

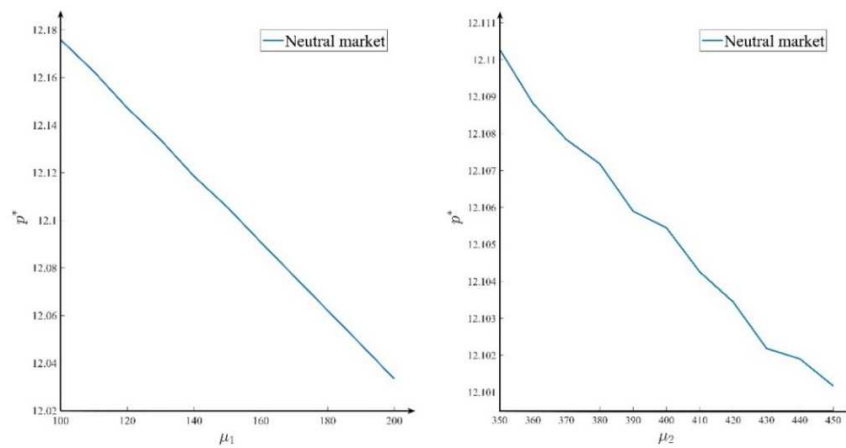


Figure 12. Effect of capacities on pricing

Observation 5. A greater service rate in the regular channel or the express channel leads to a lower DT and price differentiation and to a lower stock.

A greater drop-shipping service rate μ_1 favors the regular product. Therefore, the model reacts by increasing the regular demand which leads to quoting a shorter DT for the regular product (see Figure 10), and to reducing the stock and price for the express product (Figures 11 and 12, respectively) as this increases the number of customers switching from the express channel to the regular one. Less intuitive is the fact that a greater inventory refilling service rate μ_2 also leads to a shorter DT for the regular product (Figure 10). The reason is that a greater μ_2 allows to serve more express customers and, thus, reduces the number of customers switching to the regular channel, which enables to quote a shorter DT for this channel and attract more new customers. A greater μ_2 reduces the stockout probability which allows to keep a lower stock (see Figure 11) and thus, reduces the inventory cost. It also enables to serve more express demand which explains the decrease in the express product's price (Figure 12).

The general model confirms that a higher S-b-S leads to a greater time differentiation, which is consistent with the previous findings. The main new result is that the S-b-S does not impact the price differentiation. In practice, many retailers rely on low-cost distant suppliers for drop-shipping. We show that this is an efficient strategy for retailers facing a high S-b-S rate since selecting a fast drop-shipper implies a low time difference between express and regular products. In general, our study demonstrates that retailers must account for the S-b-S as this latter governs time differentiation and inventory strategies.

7. Conclusion

We studied the problem of a retailer serving a time- and price-sensitive market with two substitutable products that differ in the guaranteed DT and price. The express product is delivered from the stock, whereas the regular product is delivered directly from the supplier. This dual-channel distribution requires operating a hybrid system with a mix of DFS and drop-shipping. In case of stockout, some customers initially interested in the express product may switch to the regular product, which is referred to as S-b-S behavior.

When the characteristics of the express product (price and stock) are fixed and the DT of the regular product is a decision variable, we solved the model analytically and found that it is not always optimal to adopt an extreme time differentiation strategy (maximum or minimum differentiation), but a medium time differentiation could be preferred. An increase in the stock level leads the retailer to offering the regular product with a shorter DT when the retailer adopts a minimum time differentiation and with a longer DT in all other cases. When the DT and the stock level are decision variables, we distinguished two situations. In case without S-b-S, we solved the model analytically and found that there exists a threshold value for the express product's price below which it is optimal to offer both products with a minimum time differentiation and above which only the express product should be offered. Hence, the medium time differentiation cannot be optimal in this case. The consideration of the S-b-S renders the model much more complex. We therefore used

numerical experiments to derive the results. We characterized the situations where it is optimal to offer both products. In this case, we provided a near-optimal solution and showed that ignoring the S-b-S by quoting the optimal DT obtained without S-b-S leads to a considerable loss. Finally, we studied numerically the general model where the DT, stock, and price are decision variables. We used the previous results to develop a quasi-equivalent formulation of the general model with the price as a unique variable. Based on numerical experiments, we found that a higher S-b-S rate leads to a greater time differentiation, which is consistent with the findings of both previous models. In addition, a higher S-b-S leads to holding less stock but would not impact the price differentiation.

Our work can be extended to consider inventory capacity constraints in the retailer's warehouse as this may prevent the model from excessively increasing the time differentiation to favor the express demand. A future work can investigate a more complex SC where a customer order may be delivered from the stock or from several suppliers. The retailer can quote a specific DT per supplier and, consequently, offer a menu of DTs to the customers or quote a unique DT from all suppliers, which raises the issue of allocating orders to suppliers. Finally, the retailer may concentrate only on the product delivered from stock and act as a marketplace for the manufacturer. In this case, there are two independent decision-makers, the retailer who decides about the express product and the manufacturer who decides about the regular product sold through the retailer's channel.

References

- Albana, A. S., Y. Frein, R. Hammami. 2018. Effect of a lead time-dependent cost on lead time quotation, pricing, and capacity decisions in a stochastic make-to-order system with endogenous demand. *International Journal of Production Economics*, 203, 83–95.
- Ayanso, A., M. Diaby, S. K. Nair. 2006. Inventory rationing via drop-shipping in Internet retailing: A sensitivity analysis. *European Journal of Operational Research*, 171 (1), 135-152.
- Boyaci, T., S. Ray. 2003. Product differentiation and capacity cost interaction in time and price sensitive markets. *Manufacturing & Service Operations Management*, 5 (1), 18–36.
- Boyaci, T., S. Ray. 2006. The impact of capacity costs on product differentiation in delivery time, delivery reliability, and price. *Production and Operations Management*, 15 (2), 179–197.
- Hagiu, A., J. Wright. 2015. Market place or reseller? *Management Science*, 61 (1), 184-203.
- Hammami, R., Y. Frein, A. S. Albana. 2020a. Delivery time quotation and pricing in two-stage supply chains: Centralized decision-making with global and local managerial approaches. *European Journal of Operational Research*, 286 (1), 164-177.
- Hammami, R., Y. Frein, A. S. Albana. 2020b. Customer rejection to guide lead time quotation and pricing decisions. *Journal of the Operational Research Society*, DOI: 10.1080/01605682.2020.1718556.
- Hua, G., S. Wang, T. C. E. Cheng. 2010. Price and lead time decisions in dual-channel drop-shipper chains. *European Journal of Operational Research*, 205, 113-126.
- Khouja, M. 2001. The evaluation of drop shipping option for e-commerce retailers. *Computers & Industrial*

Engineering, 41, 109-126.

Khouja, M., A. C. Stylianou. 2009. A (Q, R) inventory model with a drop-shipping option for e-business. *Omega*, 37, 896-908.

Li, G., X. Zhang, M. Liu. 2019. E-tailer's procurement strategies for drop-shipping: Simultaneous vs. sequential approach to two manufacturers. *Transportation Research Part E: Logistics and Transportation Review*, 130, 108-127.

Liu, L., M. Parlar, S.X. Zhu. 2007. Pricing and lead time decisions in decentralized supply chains. *Management Science*, 53, 713-725.

Ma, S., Z. Jemai. 2019. Inventory rationing for the News-Vendor problem with a drop-shipping option. *Applied Mathematical Modelling*, 71, 438-451.

Modak, N.M., P. Kelle. 2019. Managing a dual-channel supply chain under price and delivery-time dependent stochastic demand. *European Journal of Operational Research*, 272 (1), 147-161.

Palaka, K., S. Erlebacher, D. H. Kropp. 1998. Lead-time setting, capacity utilization, and pricing decisions under lead-time dependent demand. *IIE Transactions*, 30 (2), 151-163.

Pekgün, P., P. M. Griffin, P. Keskinocak. 2008. Coordination of marketing and production for price and leadtime decisions. *IIE Transactions*, 40(1), 12-30.

Pekgün, P., P. M. Griffin, P. Keskinocak. 2017. Centralized versus Decentralized Competition for Price and Lead-Time Sensitive Demand. *Decision Sciences*, 48(6), 1198-1227.

Rabinovich, E. M. Rungtusanatham, T. M. Laseter. 2008. Physical distribution service performance and Internet retailer margins: The drop-shipping context. *Journal of Operations Management*, 26, 767-780.

Rafiei, H., M. Rabbani, 2012. Capacity coordination in hybrid make-to-stock/make-to-order production environments, *International Journal of Production Research*, 50(3), 773-789.

Rajagopalan, S. 2002. Make to order or make to stock: model and application. *Management Science*, 48(2): 241-256.

Tian, L., A. J. Vakharia, Y. R. Tan, Y. Xu. 2018. Marketplace, reseller, or hybrid: Strategic analysis of an emerging e-commerce model. *Production and Operations Management*, 27 (8), 1595-1610.

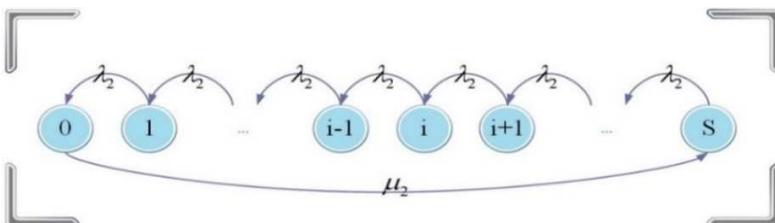
Zhao, X., K. E. Stecke, A. Prasad. 2012. Lead time and price quotation mode selection: Uniform or differentiated? *Production and Operations Management*, 21 (1), 177-193.

Zhu, S. X., (2015). Integration of capacity, pricing, and lead-time decisions in a decentralized supply chain. *International Journal of Production Economics*, 164, 14-23.

Appendix: Proofs

Calculation of stockout probability ψ_0

If the inventory is refilled to S when it is empty, the different inventory states are illustrated in the following figure.



Let's define ψ_j which stands for the probability of system in state j (i.e., probability of having j items in the stock). For each state j , the rate out of state j is equal to rate into state j . Therefore, we have for state 0, $\mu_2\psi_0 = \lambda_2\psi_1 \leftrightarrow \psi_1 = \frac{\mu_2}{\lambda_2}\psi_0$. For state $j \in [1, S-1]$, $\lambda_2\psi_j = \lambda_2\psi_{j+1} \leftrightarrow \psi_j = \psi_{j+1}$. For state S , $\mu_2\psi_S = \lambda_2\psi_S \leftrightarrow \psi_S = \frac{\mu_2}{\lambda_2}\psi_0$. The summation of all states probabilities must be equal to 1. Since $\sum_{j=0}^S \psi_j = 1$, we finally obtain $\psi_0 = \left(1 + S \frac{\mu_2}{\lambda_2}\right)^{-1}$.

Proof of Lemma 1. For $\phi=0$, constraint (8) is equivalent to $\beta_1 l^2 - (a_{1,p} - \mu_1)l - z \geq 0$. The quadratic equation $\beta_1 l^2 - (a_{1,p} - \mu_1)l - z = 0$ has a positive discriminant and only one positive root, $l_0 = \frac{a_{1,p} - \mu_1 + \sqrt{(a_{1,p} - \mu_1)^2 + 4\beta_1 z}}{2\beta_1}$. Thus, for $\phi=0$, constraint (8) is equivalent to $l \geq l_0$.

For $\phi > 0$, we obtain by standard calculus that constraint (8) is equivalent to $f_\phi(l) \geq 0$. $f_\phi(l)$ is increasing in l . Indeed, a unitary increase in l leads to a loss of β_1 regular customers and to a gain of β_2 express customers. Given that $\beta_1 > \beta_2$ and that only a part of β_2 will switch to the regular products (in case of stockout), we deduce that a unitary increase in l leads to a lower regular demand and consequently, to a higher probability of satisfying the service constraint. This means that $f_\phi(l)$ is increasing in l . Furthermore, $f_\phi(l)$ is decreasing in ϕ because a greater ϕ implies a higher percentage of express demand transformed into regular demand in the case of stockout and, thus, a higher regular demand, which implies a lower probability of satisfying the service constraint. Hence, we deduce that $f_\phi(l_0) \leq f_0(l_0)=0$. Therefore, $f_\phi(l)$ increases from a negative value to a positive one over $\left[l_0, \frac{a_{1,p}}{\beta_1}\right]$ (note that we must have $f_\phi\left(\frac{a_{1,p}}{\beta_1}\right) > 0$ since otherwise the model is infeasible). It is then deduced that $f_\phi(l)$ has only one root in $\left[l_0, \frac{a_{1,p}}{\beta_1}\right]$ (denoted by l_ϕ) and that constraint (8) is equivalent to $l \geq l_\phi$.

Proof of Lemma 2. We obtain by standard calculus $\frac{\partial^2 \pi(l)}{\partial l^2} = -\frac{2(\beta_2 \mu_2 S)^2 (m_2 - \phi m_1)}{(a_{2,p} + \beta_2 l + \mu_2 S)^3} \leq 0$ (since $m_2 \geq \phi m_1$ and $a_{2,p} + \beta_2 l + \mu_2 S \geq 0$). Thus, $\pi(l)$ is concave. After simplification, the first derivative condition is equivalent to $\beta_2^2 l^2 + 2(a_{2,p} + \mu_2 S)\beta_2 l + (a_{2,p} + \mu_2 S)^2 - \frac{\beta_2(\mu_2 S)^2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)} = 0$. This quadratic equation has a positive discriminant, $\frac{4\beta_2^3(\mu_2 S)^2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)}$, and only one positive root, $\frac{-a_{2,p} + \mu_2 S \left(\sqrt{\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)}} - 1\right)}{\beta_2}$. Therefore, $\pi(l)$ reaches its maximum in $l_{th} = \frac{-a_{2,p} + \mu_2 S \left(\sqrt{\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)}} - 1\right)}{\beta_2}$. Furthermore, we must have $l \in [l_{min}, l_{max}]$. The result follows immediately.

Proof of Proposition 1. Case 1. It can be verified that $\frac{m_2}{m_1} \leq \frac{\beta_1}{\beta_2}$ is equivalent to $\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)} \leq 1$. In this case, $l_{th} = \frac{-a_{2,p} + \mu_2 S \left(\sqrt{\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)}} - 1\right)}{\beta_2} \leq \frac{-a_{2,p}}{\beta_2}$, implying that $l_{th} \leq l_{min} = \max\{l_\phi, \frac{-a_{2,p}}{\beta_2}\}$. Consequently, $l^* = l_{min}$ (according to Lemma 2), which means that the firm opts for a minimum time-differentiation.

Case 2. For $\frac{\beta_1}{\beta_2} < \frac{m_2}{m_1} \leq \frac{1}{r_2^2} \left(\frac{\beta_1}{\beta_2} - \phi(1 - r_2^2)\right)$, it is clear that $l_{th} \leq l_{max}$ but l_{th} is not necessarily smaller than l_{min} . Hence, $l^* = \max\{l_{min}, l_{th}\}$. $l_{th} \geq l_{min} \Leftrightarrow \mu_2 \left(\sqrt{\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi\beta_2)}} - 1\right) S \geq \beta_2 l_\phi + a_{2,p}$. For small values of S (e.g., S close

to 0), we have $\mu_2 \left(\sqrt{\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi \beta_2)}} - 1 \right) S < \beta_2 l_\phi + a_{2,p}$, which means that $l_{th} < l_{min}$ and consequently $l^* = l_{min}$. When S increases, we prove in the note below that the right-hand-side, $\beta_2 l_\phi + a_{2,p}$, decreases (or remains constant for $\phi = 0$), while the left-hand-side, $\mu_2 \left(\sqrt{\frac{\beta_2(m_2 - \phi m_1)}{m_1(\beta_1 - \phi \beta_2)}} - 1 \right) S$, clearly increases. Hence, there is a threshold value of S above which we always have $l_{th} \geq l_{min}$ and consequently $l^* = l_{th}$.

Case 3. It can be verified that $\frac{m_2}{m_1} > \frac{1}{r_2^2} \left(\frac{\beta_1}{\beta_2} - \phi(1 - r_2^2) \right)$ is equivalent to $l_{th} > \frac{(1-r_2)\mu_2 S - r_2 a_{2,p}}{r_2 \beta_2}$, which means that $l_{th} > l_{max} = \min\left\{ \frac{(1-r_2)\mu_2 S - r_2 a_{2,p}}{r_2 \beta_2}, \frac{a_{1,p}}{\beta_1} \right\}$. This implies that $l^* = l_{max}$ (according to Lemma 2). Thus, the firm opts for a maximum time-differentiation.

Note. For $\phi > 0$, l_ϕ (and consequently, $\beta_2 l_\phi + a_{2,p}$) decreases in S . Indeed, constraint (8) is equivalent to $f_\phi(l) \geq 0$. Since the left-hand-side of constraint (8) is increasing in S , $f_\phi(l)$ also increases in S . In addition, $f_\phi(l)$ increases in l (as demonstrated earlier). Thus, the greater S is, the faster $f_\phi(l)$ reaches 0. Hence, l_ϕ is decreasing in S . For $\phi = 0$, it is clear that l_0 does not depend on S .

Proof of Proposition 2. If the firm adopts a minimum time-differentiation strategy, then $l^* = l_\phi$. It has been proved earlier that l_ϕ is decreasing in S . Otherwise, $l^* = l_{th}$ or l_{max} . In both cases, it is clear that l^* increases in S .

Proof of Proposition 3. If the firm adopts the maximum time-differentiation, then $l^* = l_{max}$, which does not depend on ϕ . Otherwise, $l^* = l_{th}$ or l_ϕ . It is clear that l_{th} is increasing in ϕ . As for l_ϕ , it has been shown earlier that $f_\phi(l)$ is increasing in l and decreasing in ϕ for a given l . Therefore, l_ϕ , which is the root of $f_\phi(l) = 0$, gets bigger when ϕ increases.

Proof of Lemma 3. If $\phi = 0$, then we always have $S_1(l) \leq 0$, implying that $S^*(l) = \max\{S_{th}(l), S_2(l)\}$. By standard calculus, it comes that $S^*(l) = S_2(l)$ if $m_2 \leq \frac{h}{(1-r_2)^2 \mu_2}$ and $S^*(l) = S_{th}(l)$ otherwise. Furthermore, constraint (11) is equivalent to $l \geq l_0 = \frac{a_{1,p} - \mu_1 + \sqrt{(a_{1,p} - \mu_1)^2 + 4\beta_1 z}}{2\beta_1}$. Since, in addition, $\frac{-a_{2,p}}{\beta_2} \leq l \leq \frac{a_{1,p}}{\beta_1}$ (to guarantee positive demand for each product), model $(M_{l,S})$ with $\phi = 0$ is equivalent to the single-variable model of Lemma 3.

Proof of Lemma 4. This result is obtained by using the first derivative condition with the linear functions defining model $(M_{l,S})$ and given in Lemma 3. In each case (i.e., $m_2 \leq \frac{h}{(1-r_2)^2 \mu_2}$ or $m_2 > \frac{h}{(1-r_2)^2 \mu_2}$), the shortest DT is optimal when the first derivative is negative (i.e., when $m_2 \leq \frac{m_1 \beta_1}{r_2 \beta_2} + \frac{h}{(1-r_2)\mu_2}$ or $m_2 \leq \left(\sqrt{\frac{m_1 \beta_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}} \right)^2$, respectively), otherwise the longest DT is optimal. As explained in the proof of Lemma 3, $S^*(l) = S_2(l)$ if $m_2 \leq \frac{h}{(1-r_2)^2 \mu_2}$ and $S^*(l) = S_{th}(l)$ otherwise.

Proof of Proposition 4. We let cases 1, 2, 3 and 4 respectively denote the situations where $m_2 \leq \frac{h}{(1-r_2)^2 \mu_2}$ and $m_2 \leq \frac{\beta_1 m_1}{r_2 \beta_2} + \frac{h}{(1-r_2)\mu_2}$, $m_2 \leq \frac{h}{(1-r_2)^2 \mu_2}$ and $m_2 > \frac{\beta_1 m_1}{r_2 \beta_2} + \frac{h}{(1-r_2)\mu_2}$, $m_2 > \frac{h}{(1-r_2)^2 \mu_2}$ and $m_2 \leq \left(\sqrt{\frac{\beta_1 m_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}} \right)^2$, and finally $m_2 > \frac{h}{(1-r_2)^2 \mu_2}$ and $m_2 > \left(\sqrt{\frac{\beta_1 m_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}} \right)^2$. Note that $\frac{\beta_1 m_1}{r_2 \beta_2} + \frac{h}{(1-r_2)\mu_2} < \frac{h}{(1-r_2)^2 \mu_2} \Leftrightarrow \frac{\beta_1 m_1}{\beta_2} < \frac{h}{\mu_2} \left(\frac{r_2}{1-r_2} \right)^2$ and that $\left(\sqrt{\frac{\beta_1 m_1}{\beta_2}} + \sqrt{\frac{h}{\mu_2}} \right)^2$

$\sqrt{\frac{h}{\mu_2}} > \frac{h}{(1-r_2)^2 \mu_2} \Leftrightarrow \frac{\beta_1 m_1}{\beta_2} > \frac{h}{\mu_2} \left(\frac{r_2}{1-r_2} \right)^2$. Therefore, cases 2 and 3 cannot be feasible simultaneously. Thus, varying the value of p enables to have one of the two following sequences: case 1-case 2-case 4 or case 1-case 3-case 4. Using the optimal solutions provided in Lemma 4, the result follows immediately.

Proof of Proposition 5. When both products are offered to the customers, it has been demonstrated that $l^* = l_0 = \frac{a_{1,p} - \mu_1 + \sqrt{(a_{1,p} - \mu_1)^2 + 4\beta_1 z}}{2\beta_1}$. Since $a_{1,p} = A - \alpha_1 p_1 + \alpha_2 p$, it is deduced that the higher the value of p is, the longer l^* becomes. As for the stock, we have in this case $S^*(l) = \max\{S_{th}(l), S_2(l)\}$ (see the proof of Lemma 3). Consequently, S^* is increasing in λ_2 (since both of $S_{th}(l)$ and $S_2(l)$ are increasing in λ_2). Given in addition that λ_2 is increasing in l and that the optimal DT is increasing in p , we deduce that the higher the value of p is, the greater S^* becomes.

Proof of Lemma 5. For a given l , we have established that $S^*(l) = \max\{S_{th}(l), S_1(l), S_2(l)\}$, where $S_{th}(l) = \frac{\lambda_2}{\mu_2} \left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} - 1 \right)$, $S_1(l) = \frac{\lambda_2}{\mu_2} \left(\frac{\phi \lambda_2}{\mu_1 - \lambda_1 - \frac{z}{l}} - 1 \right)$, and $S_2(l) = \frac{r_2 \lambda_2}{1 - r_2 \mu_2}$.

It is firstly noted that $S_{th}(l) \geq S_1(l)$ is equivalent to $l \geq l_1$ where $l_1 = \frac{\left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} (a_{1,p} - \mu_1) + \phi a_{2,p} \right) + \sqrt{\left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} (a_{1,p} - \mu_1) + \phi a_{2,p} \right)^2 + 4 \sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} \left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} \beta_1 - \phi \beta_2 \right) z}}{2 \left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} \beta_1 - \phi \beta_2 \right)}$. Indeed, $S_{th}(l) \geq$

$S_1(l) \Leftrightarrow \left(\beta_1 \sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} - \phi \beta_2 \right) l^2 - \left(\sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} (a_{1,p} - \mu_1) + \phi a_{2,p} \right) l - z \sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} \geq 0$. Given that

$\beta_1 \sqrt{\frac{\mu_2(m_2 - \phi m_1)}{h}} - \phi \beta_2 > 0$ (since $\frac{\mu_2(m_2 - \phi m_1)}{h} > 1$ and $\beta_1 > \beta_2$), the discriminant of this quadratic function is positive.

Hence, $S_{th}(l) \geq S_1(l) \Leftrightarrow l \geq l_1$.

In addition, it can be demonstrated by standard calculus that $S_2(l) \geq S_1(l)$ is equivalent to $l \geq l_2 =$

$$\frac{a_{1,p} + \phi(1-r_2)a_{2,p} - \mu_1 + \sqrt{(a_{1,p} + \phi(1-r_2)a_{2,p} - \mu_1)^2 + 4(\beta_1 - \phi(1-r_2)\beta_2)z}}{2(\beta_1 - \phi(1-r_2)\beta_2)}, \text{ and } S_2(l) \geq S_{th}(l) \text{ is equivalent to } m_2 - \phi m_1 \leq \frac{h}{(1-r_2)^2 \mu_2}.$$

Substituting $S^*(l)$ by its expression in each situation, we get the following equivalent formulation of model $(M_{l,S})$.

-If $m_2 - \phi m_1 \leq \frac{h}{(1-r_2)^2 \mu_2}$, then the model is equivalent to:

$$\text{Max } \pi(l) = \begin{cases} \left(m_1 - \frac{m_2}{\phi} \right) \mu_1 + m_2 a_{2,p} + \frac{m_2 a_{1,p}}{\phi} + \frac{h a_{2,p}}{\mu_2} - \left(m_1 - \frac{m_2}{\phi} \right) \frac{z}{l} + \left(m_2 \beta_2 - \frac{m_2 \beta_1}{\phi} + \frac{h \beta_2}{\mu_2} \right) l & \text{if } l_0 \leq l \leq l_2 \\ - \left(\frac{h \phi}{\mu_2} \right) \frac{(a_{2,p} + \beta_2 l)^2}{\mu_1 - a_{1,p} + \beta_1 l - \frac{z}{l}} & \\ m_1(a_{1,p} - \beta_1 l) + \left((1-r_2)\phi m_1 + r_2 m_2 - \frac{h r_2}{(1-r_2)\mu_2} \right) (a_{2,p} + \beta_2 l) & \text{if } l_2 \leq l \leq \frac{a_{1,p}}{\beta_1} \end{cases}$$

- If $m_2 - \phi m_1 > \frac{h}{(1-r_2)^2 \mu_2}$, then the model is equivalent to:

$$\text{Max } \pi(l) = \begin{cases} \left(m_1 - \frac{m_2}{\phi} \right) \mu_1 + m_2 a_{2,p} + \frac{m_2 a_{1,p}}{\phi} + \frac{h a_{2,p}}{\mu_2} - \left(m_1 - \frac{m_2}{\phi} \right) \frac{z}{l} + \left(m_2 \beta_2 - \frac{m_2 \beta_1}{\phi} + \frac{h \beta_2}{\mu_2} \right) l \\ - \left(\frac{h \phi}{\mu_2} \right) \frac{(a_{2,p} + \beta_2 l)^2}{\mu_1 - a_{1,p} + \beta_1 l - \frac{z}{l}} & \text{if } l_0 \leq l \leq l_1 \\ m_1 (a_{1,p} - \beta_1 l) + \left(\frac{m_2 \mu_2 + h - 2\sqrt{\mu_2 h (m_2 - \phi m_1)}}{\mu_2} \right) (a_{2,p} + \beta_2 l) & \text{if } l_1 \leq l \leq \frac{a_{1,p}}{\beta_1} \end{cases}$$

By introducing the notation $\theta = \max \left(\sqrt{\frac{\mu_2 (m_2 - \phi m_1)}{h}}, \frac{1}{1 - r_2} \right)$ and $l_\theta =$

$\frac{(\theta(a_{1,p} - \mu_1) + \phi a_{2,p}) + \sqrt{(\theta(a_{1,p} - \mu_1) + \phi a_{2,p})^2 + 4\theta(\theta\beta_1 - \phi\beta_2)z}}{2(\theta\beta_1 - \phi\beta_2)}$, we get the compact formulation of model ($M_{l,S}$) given in Lemma 5.

Simulation of the expected stock level with $ROP > 0$. We consider different values of λ_2 , μ_2 and S as well as different values of ROP ($ROP=0$, $ROP=1$, $ROP=2$, $ROP=5\%$ of S , and $ROP=10\%$ of S). For each value of ROP , we simulate the system with ARENA for the equivalent of 300,000 hours and calculate the expected stock level. The results are given in the table below. We also give between parentheses the exact stock level obtained analytically for $ROP=0$ (i.e., $\bar{S} = \frac{S+1}{2} \frac{1}{1 + \frac{\lambda_2}{\mu_2 S}}$) and for $ROP=1$ (i.e., $\bar{S} = \frac{(S-1)(\frac{S\mu(\mu+\lambda)+2\mu^2}{2})}{\mu^2 + \lambda^2 + \mu\lambda + \mu(\mu+\lambda)(S-2)}$). We then compare the expected stock level obtained with simulation (and analytically when possible) to $\frac{S}{2}$. We see that considering $\bar{S} \approx \frac{S}{2}$ is a very good approximation for the expected stock level.

λ_2	μ_2	S	$\frac{\lambda_2}{\mu_2 S}$	$\frac{S}{2}$	$ROP=0$	$ROP=1$	$ROP=2$	$ROP=5\%$ of S	$ROP=10\%$ of S
80	40	60	0.03	30	(29.51) 29.48	(29.66) 29.64	29.94	30.28	31.58
80	40	80	0.02	40	(39.51) 39.50	(39.66) 39.60	39.96	40.76	42.52
80	60	60	0.02	30	(29.83) 29.80	(30.04) 30.06	30.33	30.82	32.18
80	60	80	0.01	40	(39.83) 39.82	(40.04) 39.97	40.40	41.26	43.22
80	80	60	0.01	30	(30.00) 30.02	(30.24) 30.29	30.59	31.08	32.52
80	80	80	0.01	40	(40.00) 39.97	(40.24) 40.30	40.64	41.49	43.44
100	40	60	0.04	30	(29.28) 29.29	(29.32) 29.35	29.63	29.90	31.14
100	40	80	0.03	40	(39.27) 39.26	(39.39) 39.40	39.53	40.44	42.17
100	60	60	0.02	30	(29.67) 29.62	(29.84) 29.82	30.14	30.55	31.84